Phys 396 2018

Lecture 27

Term paper:

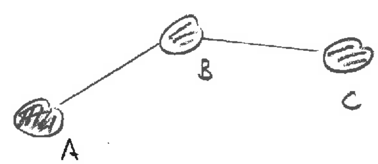
Final presentation: ~~Tues~~ Tues 11 Dec 1-3 pm

(about 15 minutes each)

Gates for spin-½ qubits

We consider a molecule consisting of distinct spin-½ nuclei, each of which represents a distinct qubit. These are placed in a constant magnetic field

$$\vec{B} = B_0 \hat{z}$$



The resulting Hamiltonian for any single spin is:

$$\hat{H} = \frac{\hbar \omega_0}{2} \hat{\sigma}_z$$

where ω_0 is proportional to B_0 and other quantities that only depend on the nucleus and its immediate environment. It follows that if the system is allowed to evolve for time t then the unitary evolution operator is

$$\hat{U}(t) = e^{-i\omega_0 t \hat{\sigma}_z/2}$$

This is a rotation about the z-axis through angle $\omega_0 t$.

Thus

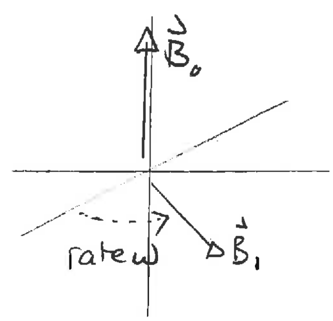
An arbitrary rotation about the z-axis can be attained by letting the system evolve for the appropriate time t under $\vec{B} = B_0 \hat{z}$

To produce a rotation about a perpendicular axis we supply an additional rotating magnetic field

$$\vec{B}_1 = B_1 \left[\cos \omega t \, \hat{x} + \sin \omega t \, \hat{y} \right]$$

and let the system evolve under $\vec{B} = \vec{B}_0 + \vec{B}_1$

By moving to the rotating frame we can eventually show that the unitary evolution operator is



$$\hat{U}(t) = e^{-i\omega t \hat{\sigma}_z/2} \, e^{-i \hat{H}_{rot} t/\hbar}$$

where

$$\hat{H}_{rot} = \frac{\hbar}{2}(\omega_0 - \omega)\hat{\sigma}_z + \frac{\hbar \omega_1}{2} \hat{\sigma}_x$$

and ω_1 is proportional to B_1. So

$\omega_0 \rightarrow$ strength of $\vec{B}_0$

$\omega_1 \rightarrow$ " " $\vec{B}_1$

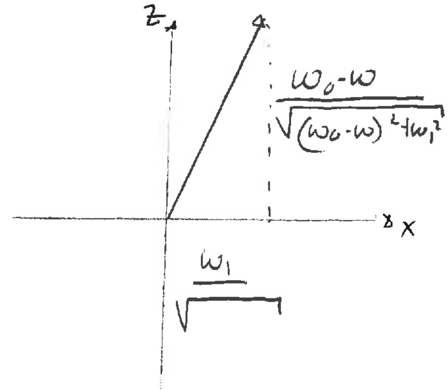
$\omega \rightarrow$ rate at which $\vec{B}_1$ rotates

Then the rightmost term in $\hat{U}(t)$ is:

$$e^{-i\left[(\omega_0-\omega)\hat{\sigma}_z + \omega_1 \sigma_x\right]t/2}$$

which is a rotation about an axis in the x-z plane. Specifically with $(\omega_0-\omega)\hat{\sigma}_z + \omega_1\hat{\sigma}_x = \left[(\omega_0-\omega)\hat{\sigma}_z + \omega_1\hat{\sigma}_x\right]\dfrac{\sqrt{(\omega_0-\omega)^2+\omega_1^2}}{\sqrt{}}$

we see that it is a rotation about the illustrated unit vector through angle $\sqrt{(\omega_0-\omega)^2+\omega_1^2}\; t$.



We can therefore engineer a rotation about $\hat{x}$ by adjusting $\omega = \omega_0$.

Thus we get

$$\hat{U}(t) = \underbrace{e^{-i\omega_0 t\hat{\sigma}_z/2}}_{\text{rotation about } z \text{ through } \omega_0 t}\ \underbrace{e^{-i\omega_1 t\hat{\sigma}_x/2}}_{\text{rotation about } x \text{ through } \omega_1 t}$$

So we can do a rotation about $\hat{x}$ by:

Apply a rotating magnetic field with rotation rate $\omega = \omega_0$.

Then

$\boxed{\text{Run rotating field for time } t} \rightarrow \boxed{\text{Turn rotating field off and let spin evolve under } \vec{B}_0 \text{ only for time } t' \text{ s.t. } \omega_0(t+t') = 2\pi}$

Rot about x angle $\omega_1 t$ + Rot about z angle $\omega_0 t$ + Rot about z angle $\omega_0 t'$ $\equiv$ rot about x angle $\omega_1 t$

Thus we can get:

Rotation about $\hat{z}$ $\rightsquigarrow$ evolve under $\vec{B}_0$ only

Rotation about $\hat{x}$ $\rightsquigarrow$ evolve under rotating magnetic field $\vec{B}_1$ with frequency $\omega = \omega_0$ and also under just $\vec{B}_0$

This allows us to construct arbitrary rotations about $\hat{x}$ and $\hat{z}$ and therefore any rotation about any axis. Thus we can attain any single qubit unitary in this way.

Two qubit gates

Recall that we need two qubit gates between arbitrary pairs of qubits. If we can generate a CNOT, then we can generate <u>any</u> two qubit gate. So we focus on generating a CNOT. We will

1) first show how to construct a CNOT from a controlled-Z

2) second show how to construct a controlled-Z using a specific interaction between two spins.

1 CNOT gates from controlled-Z gates

Recall that a CNOT gate can be represented as

$$\hat{C}_{\hat{\sigma}_x} = |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_x.$$

A controlled-Z can be represented as

$$\hat{C}_{\hat{\sigma}_z} = |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_z.$$

a) Represent each of these in terms of matrices in the computational basis.

b) Show how to construct the CNOT using a controlled-Z and single qubit operations.

<u>Answer</u>: a) $|0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_x$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{C}_{\hat{\sigma}_x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then $|0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_z$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$\Rightarrow \quad \hat{C}_{\sigma_z} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

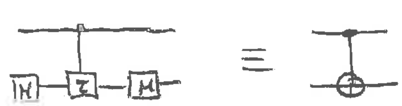
b) We need a unitary $\hat{U}$ s.t. $\hat{U}\hat{U} = \hat{I}$ and $\hat{U}\hat{\sigma}_z\hat{U} = \sigma_x$

Then

$$(\hat{I} \otimes \hat{U})(\hat{C}_{\sigma_z})(\hat{I} \otimes \hat{U}) = |0\rangle\langle 0| \otimes \hat{U}\hat{U} + |1\rangle\langle 1| \otimes \hat{U}\sigma_z\hat{U}$$

$$= |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_x$$

Clearly $\hat{H}$ works so



The way in which a two-qubit gate can be constructed requires a coupling between two spins. A specific example of this is the "J coupling" which results from inter-bond interactions. The J-coupling Hamiltonian is

$$\hat{H}_J = \frac{\hbar\pi}{2} J \; \hat{\sigma}_z \otimes \hat{\sigma}_z$$

We can then explore evolution under this for specified durations of time in conjunction with single qubit gates.

2 Evolution under J-coupling

Consider the J coupling Hamiltonian as presented in lecture.

a) Determine a matrix expression for

$$\hat{U}(t) = e^{-i\hat{H}_J t/\hbar}.$$

b) Evaluate the expression for $t = 1/2J$.

c) Evaluate the expression for $t = 1/J$.

d) Consider $\hat{U}(1/2J)$ and suggest two single bit rotations, each about the z axis such that the result is the controlled-Z gate.

Answer: a)

$$\hat{H} = \frac{\hbar \pi J}{2} \hat{\sigma}_z \otimes \hat{\sigma}_z$$

$$\hat{U}(t) = e^{-i\frac{\pi J t}{2}\hat{\sigma}_z \otimes \hat{\sigma}_z}$$

$$= \hat{I} - i\frac{\pi J t}{2}(\hat{\sigma}_z \otimes \hat{\sigma}_z) + \frac{1}{2!}\left(\frac{-i\pi J t}{2}\right)\underbrace{(\hat{\sigma}_z \times \hat{\sigma}_z)^2}_{=\hat{I}} + \dots$$

$$= \hat{I}\cos\left(\frac{\pi J t}{2}\right) - i\,\hat{\sigma}_z \otimes \hat{\sigma}_z \sin\left(\frac{\pi J t}{2}\right)$$

$$= \cos\frac{\pi J t}{2}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - i\sin\left(\frac{\pi J t}{2}\right)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\pi J t/2} & 0 & 0 & 0 \\ 0 & e^{i\pi J t/2} & 0 & 0 \\ 0 & 0 & e^{i\pi J t/2} & 0 \\ 0 & 0 & 0 & e^{-i\pi J t/2} \end{pmatrix}$$

b) $$\hat{U}\left(\tfrac{1}{2J}\right) = \begin{pmatrix} e^{-i\pi/4} & & & \\ & e^{i\pi/4} & & \\ & & e^{i\pi/4} & \\ & & & e^{-i\pi/4} \end{pmatrix} = e^{-i\pi/4} \begin{pmatrix} 1 & & & \\ & e^{i\pi/2} & & \\ & & e^{i\pi/2} & \\ & & & 1 \end{pmatrix}$$

c) $$\hat{U}\left(\tfrac{1}{J}\right) = -i\,\hat{\sigma}_z \otimes \hat{\sigma}_z = -i \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

d) Consider a rotation on the left qubit through angle α and on right by angle β. Then these correspond to

$$\hat{U}_{rot} = e^{-i\alpha\sigma_z/2} \otimes e^{-i\beta\hat{\sigma}_z/2}$$

$$= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \otimes \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i(\alpha+\beta)/2} & & & \\ & e^{-i(\alpha-\beta)/2} & & \\ & & e^{-i(\beta-\alpha)/2} & \\ & & & e^{-i(-\alpha-\beta)/2} \end{pmatrix} = e^{-i(\alpha+\beta)/2} \begin{pmatrix} 1 & & & \\ & e^{i\beta} & & \\ & & e^{i\alpha} & \\ & & & e^{i(\alpha+\beta)} \end{pmatrix}$$

Now

$$\hat{U}_{rot}\,\hat{U}\left(\tfrac{1}{2J}\right) = e^{i(\alpha+\beta)/2}\begin{pmatrix}1 & & & \\ & e^{i\beta} & & \\ & & e^{i\alpha} & \\ & & & e^{i(\alpha+\beta)}\end{pmatrix}\begin{pmatrix}1 & & & \\ & e^{i\pi/2} & & \\ & & e^{i\pi/2} & \\ & & & -1\end{pmatrix}$$

$$= \underbrace{e^{i(\alpha+\beta)/2}}_{\text{global phase}}\begin{pmatrix}1 & & & \\ & e^{i(\beta+\pi/2)} & & \\ & & e^{i(\alpha+\pi/2)} & \\ & & & e^{i(\alpha+\beta)}\end{pmatrix}$$

If $\alpha=\beta=3\pi/2$ then

$$e^{i(\beta+\pi/2)} = e^{i2\pi} = 1$$

$$e^{i(\alpha+\pi/2)} = e^{i2\pi} = 1$$

$$e^{i(\alpha+\beta)} = e^{i3\pi} = -1$$

So we have constructed the C-Z gate $\boxtimes$

Thus, to construct the controlled-X gate, we can evolve the pair of spins under the J-coupling followed by assorted single bit gates