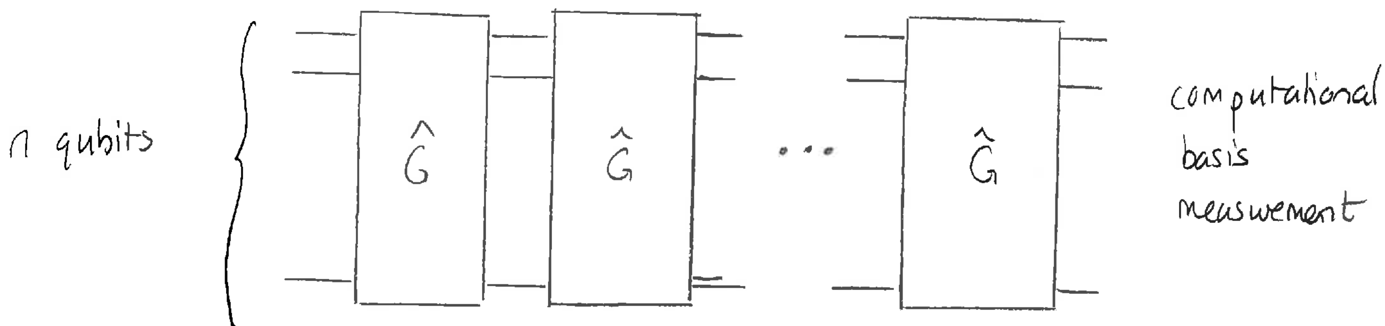
Phys 396 2018

Lecture 25

Grover's algorithm

The structure of Grovers algorithm for searching an n bit database is:



Initial state $|\Psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle_d$

Here the Grover iterate is:

$$\hat{G} = \hat{D}\,\hat{U}_f$$

This consists of the oracle unitary defined via:

$$\hat{U}_f\,|x\rangle_d = (-1)^{f(x)}\,|x\rangle_d$$

The inversion about the average operation is:

$$\hat{D} := 2|\Phi\rangle\langle\Phi| - \hat{I}$$

where

$$|\Phi\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle_d$$

This maps:

$$|\Psi\rangle = \alpha_0|0\rangle_d + \cdots + \alpha_{N-1}|N-1\rangle_d$$

$$\downarrow$$

$$\beta_0|0\rangle_d + \cdots + \beta_{N-1}|N-1\rangle_d$$

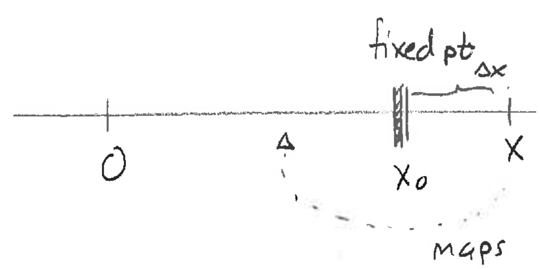
where

$$\beta_k = 2\langle\alpha\rangle - \alpha_k$$

and

$$\langle\alpha\rangle = \frac{1}{N}\sum \alpha_k$$

Note that this is called inversion about the average for the following reason. Consider a process which reflects about a fixed point on the real line



This takes

$$x \to x_0 - \Delta x$$

where $\Delta x = x - x_0$

$$\Rightarrow \quad x \to x_0 - (x - x_0) = 2x_0 - x$$

$$\Rightarrow \quad x \to 2x_0 - x$$

1 Grover's algorithm on a three bit database

Consider an unstructured search of a three bit database. Suppose that the marked item is located at $x = 7$.

a) Evaluate the state of the three qubit system after a single application of $\hat{G}$. With what probability will a computational basis measurement reveal the marked item's location?

b) Evaluate the state of the three qubit system after two applications of $\hat{G}$. With what probability will a computational basis measurement reveal the marked item's location?

c) Evaluate the state of the three qubit system after three applications of $\hat{G}$. With what probability will a computational basis measurement reveal the marked item's location?

Answer: a) Initially

$$|\Psi_0\rangle = \frac{1}{\sqrt{8}}\left\{|0\rangle + |1\rangle + \cdots + |7\rangle\right\}$$

$$\downarrow U_f$$

$$\frac{1}{\sqrt{8}}\left\{|0\rangle + |1\rangle + \cdots + |6\rangle - |7\rangle\right\}$$

$$\langle\alpha\rangle = \frac{1}{8}\left(\frac{1}{\sqrt{8}}\,7 - 1\right) = \frac{3}{4}\frac{1}{\sqrt{8}}$$

$$\alpha_k = \frac{1}{\sqrt{8}} \to \frac{6}{4}\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{8}} = \frac{1}{2}\frac{1}{\sqrt{8}}$$

$$\downarrow \hat{D}$$

$$\alpha_k = -\frac{1}{\sqrt{8}} \to \frac{6}{4}\frac{1}{\sqrt{8}} + \frac{1}{\sqrt{8}} = \frac{10}{4}\frac{1}{\sqrt{8}}$$

$$\frac{1}{\sqrt{8}}\left\{\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \cdots + \frac{1}{2}|6\rangle + \frac{5}{2}|7\rangle\right\}$$

$$|\Psi_1\rangle = \frac{1}{2\sqrt{8}}\left\{|0\rangle + |1\rangle + \cdots + |6\rangle + 5|7\rangle\right\}$$

$$\text{Prob}(0) = \text{Prob}(1) = \text{Prob}(2) = \cdots = \text{Prob}(6) = \left(\frac{1}{2\sqrt{8}}\right)^2 = \frac{1}{32}$$

$$\text{Prob}(7) = \left(\frac{5}{2\sqrt{8}}\right)^2 = \frac{25}{32} = \boxed{0.78}$$

b) $$|\Psi_1\rangle = \frac{1}{2\sqrt{8}}\left\{ |0\rangle + |1\rangle + \cdots + |6\rangle + 5|7\rangle \right\}$$

$$\downarrow \hat{U}_F$$

$$\frac{1}{2\sqrt{8}}\left\{ |0\rangle + |1\rangle + \cdots + |6\rangle - 5|7\rangle \right\}$$

$$\downarrow \hat{D}$$

$$|\Psi_2\rangle = \frac{1}{4\sqrt{8}}\left\{ -|0\rangle - |1\rangle + \cdots - |6\rangle + 11|7\rangle \right\}$$

$$\langle\alpha\rangle = \frac{1}{8}\left[\frac{7}{2\sqrt{8}} - \frac{5}{2\sqrt{8}}\right] = \frac{2}{16\sqrt{8}} = \frac{1}{8\sqrt{8}}$$

$$\langle\alpha\rangle = \frac{1}{8\sqrt{8}}$$

So

$$\alpha_t = \frac{1}{2\sqrt{8}} \to \frac{1}{4\sqrt{8}} - \frac{1}{2\sqrt{8}} = \frac{-1}{4\sqrt{8}}$$

$$\alpha_k = \frac{-5}{2\sqrt{8}} \to \frac{1}{4\sqrt{8}} + \frac{5}{2\sqrt{8}} = \frac{11}{4\sqrt{8}}$$

So $$\text{Prob}(0) = \text{Prob}(1) = \cdots = \text{Prob}(6) = \left(\frac{1}{4\sqrt{8}}\right)^2 = \frac{1}{128} = 0.0078$$

$$\text{Prob}(7) = \left(\frac{11}{4\sqrt{8}}\right)^2 = \frac{121}{128} = \boxed{0.9453}$$

c) $|\Psi_2\rangle = \frac{1}{4\sqrt{8}}\left\{-|0\rangle - |1\rangle \cdots -|6\rangle + 11|7\rangle\right\}$

$$\downarrow U_f$$

$$= -\frac{1}{4\sqrt{8}}\left\{|0\rangle + |1\rangle + \cdots + |6\rangle + 11|7\rangle\right\}$$

$$\downarrow \hat{D}$$

$$= \frac{1}{8\sqrt{8}}\left\{-7|0\rangle - 7|1\rangle \cdots -7|6\rangle + 13|7\rangle\right\}.$$

$$\langle\alpha\rangle = \frac{1}{8}\left\{-\frac{1}{4\sqrt{8}}(7+11)\right\} = -\frac{18}{32\sqrt{8}} = -\frac{9}{16\sqrt{8}}$$

$$\alpha_k = -\frac{1}{4\sqrt{8}} \rightarrow -\frac{18}{16\sqrt{8}} + \frac{1}{4\sqrt{8}} = -\frac{9}{8\sqrt{8}} + \frac{2}{8\sqrt{8}} = -\frac{7}{8\sqrt{8}}$$

$$\alpha_k = -\frac{11}{4\sqrt{8}} \rightarrow -\frac{18}{16\sqrt{8}} + \frac{11}{4\sqrt{8}} = -\frac{9}{8\sqrt{8}} + \frac{22}{8\sqrt{8}} = \frac{13}{8\sqrt{8}}$$

$$\text{Prob}(0) = \text{Prob}(1) = \ldots = \text{Prob}(6) = \left(\frac{-7}{8\sqrt{8}}\right)^2 = \frac{49}{512}$$

$$\text{Prob}(7) = \left(\frac{13}{8\sqrt{8}}\right)^2 = \frac{169}{512} = \boxed{0.33}$$

Again we see that the algorithm offers a speedup. With two oracle queries we will locate the marked item with probability 0.945.

A classical algorithm will locate the marked item with probability 0.25 after just two queries and with probability 0.50 after four queries.

We also see that going too far reduces the probability of locating the marked item

Performance of Grover's algorithm

If we consider the previous example then we can see that the coefficients of the unmarked states are the same at any step. The coefficient of the marked state differs. So initially

$$|\Psi_0\rangle = \boxed{\frac{1}{\sqrt{8}}}\left\{|0\rangle_d + |1\rangle_d + \dots + |6\rangle_d\right\} + \boxed{\frac{1}{\sqrt{8}}}|7\rangle_d$$

After the first Grover iterate:

$$|\Psi_1\rangle = \boxed{\frac{1}{2\sqrt{8}}}\left\{|0\rangle_d + |1\rangle_d + \dots + |6\rangle_d\right\} + \boxed{\frac{5}{2\sqrt{8}}}|7\rangle_d$$

After the second

$$|\Psi_2\rangle = \boxed{\frac{-1}{4\sqrt{8}}}\left\{|0\rangle_d + |1\rangle_d + \dots + |6\rangle_d\right\} + \boxed{\frac{11}{4\sqrt{8}}}|7\rangle_d$$

In order to track the algorithms progress we need only follow the highlighted coefficients. The state that accompanies these is less important. This will help assess the number of iterations required for success.

Suppose that we search an n qubit database with one marked item. Without loss of generality suppose that the marked item location is $N-1$.

Then we construct the states:

$$|\psi_{um}\rangle = \frac{1}{\sqrt{N-1}}\left\{|0\rangle_d + |1\rangle_d + \dots + |N-2\rangle_d\right\}$$

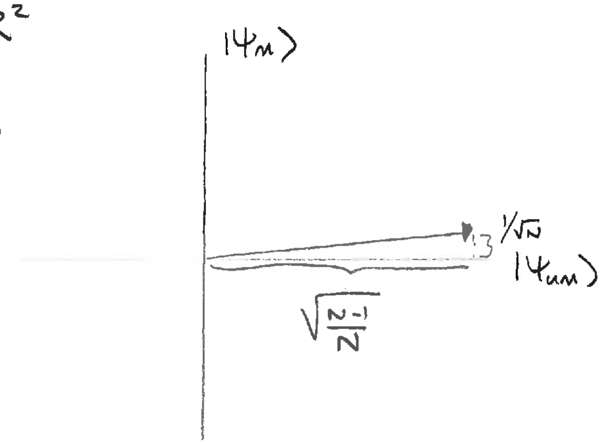
$$|\psi_m\rangle = |N-1\rangle_d.$$

These are both normalized and orthogonal. So initially

$$|\Psi_0\rangle = \frac{\sqrt{N-1}}{\sqrt{N}}|\psi_{um}\rangle + \frac{1}{\sqrt{N}}|\psi_m\rangle$$

$$\Rightarrow \boxed{|\Psi_0\rangle = \sqrt{\frac{N-1}{N}}\,|\psi_{um}\rangle + \frac{1}{\sqrt{N}}|\psi_m\rangle}$$

Throughout the algorithm the coefficients are real and so we can visualize this as a vector in $\mathbb{R}^2$



What do the two operations in $\hat{G}$ do to this.

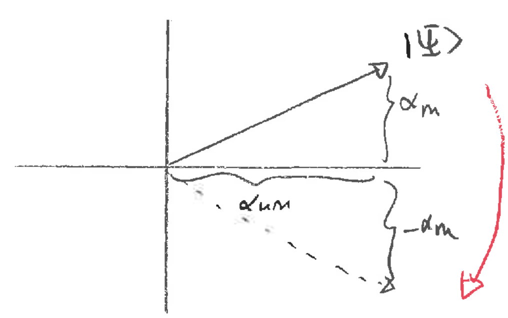
Consider a general vector:

$$|\Psi\rangle = \alpha_{um}|\psi_{um}\rangle + \alpha_m|\psi_m\rangle$$

The oracle maps this to

$$\hookrightarrow \alpha_{um}|\psi_{um}\rangle - \alpha_m|\psi_m\rangle$$

This is a reflection about the horizontal axis.



Now consider the inversion about the average

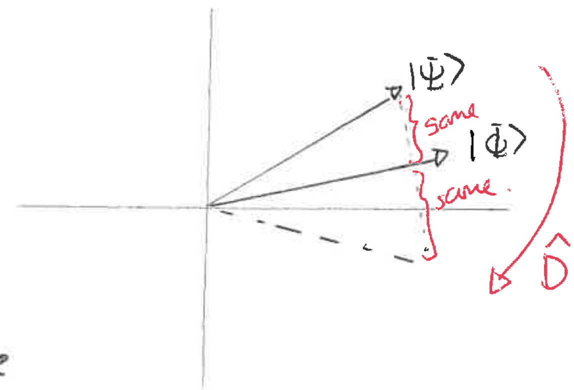
$$\hat{D} = 2|\Phi\rangle\langle\Phi| - \hat{I}$$

We can show that this is a reflection about the vector

$$|\Phi\rangle = \frac{1}{\sqrt{N}} \sum |x\rangle_d$$

$$= \sqrt{\frac{N-1}{N}}\, |\psi_{um}\rangle + \frac{1}{\sqrt{N}}\, |\psi_m\rangle$$

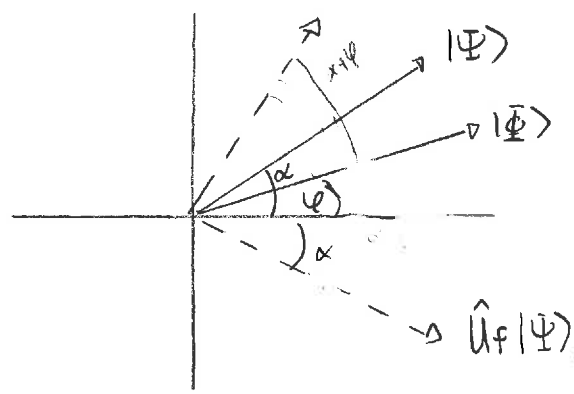
So $\hat{G}$ is a product of two reflections, which is also a rotation



Let's analyze this geometrically. Suppose that the angle between $|\Phi\rangle$ and the x-axis is φ. Suppose that the angle to $|\Psi\rangle$ is α

Then $\hat{U}_f$ maps $|\Psi\rangle$ to a vector at angle α below the horiz axis

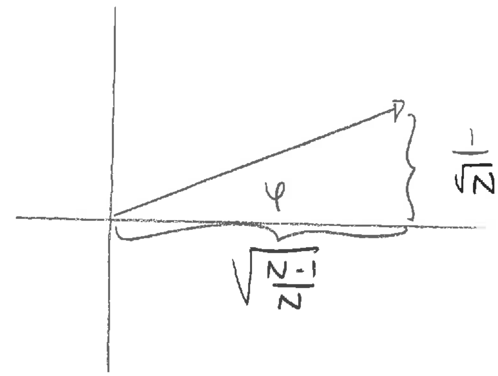
Then $\hat{D}$ maps $\hat{U}_f|\Psi\rangle$ to a vector at angle $\varphi+\alpha$ above $|\Phi\rangle$ That is an angle of $\alpha+2\varphi$ above horiz.



Thus the iterate rotates the state $|\Psi\rangle$ through angle 2φ toward the vertical axis

The initial state is exactly at angle φ if we rotate k times the final state will be at angle $k2\varphi + \varphi$ from the horiz axis. We want this equal to $\pi/2$. Thus we need

$$k \text{ s.t. } (2k+1)\varphi = \pi/2$$



Now we can get φ via:

$$\tan\varphi = \frac{1/\sqrt{N}}{\sqrt{N-1}/\sqrt{N}}$$

$$\Rightarrow \tan\varphi = \frac{1}{\sqrt{N-1}}.$$

$$\Rightarrow \varphi = \arctan\left(\frac{1}{\sqrt{N-1}}\right)$$

Then if $N \gg 1$, $\sqrt{N-1} \gg 1$ and $\varphi \approx \frac{1}{\sqrt{N-1}}$

Thus we need k iterations s.t.

$$2k+1 = \frac{\pi}{2\varphi} \approx \frac{\pi}{2}\sqrt{N-1} \approx \frac{\pi}{2}\sqrt{N}$$

$$\Rightarrow k \approx \frac{\pi}{4}\sqrt{N}$$

So we need about $\sqrt{N}$ iterations and thus $\sqrt{N}$ oracle queries.