Lecture 22

Tues: Exam

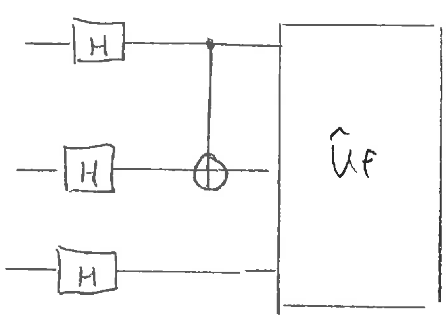
Covers: Lecture 14 – 22
HW 6 – 10

* Unitary evolution of states
* Evolution operators
* Gates
* Oracle unitaries
* Unitaries / gates acting on states

Bring: Second half ~~sheet~~ or letter sheet single side.

Quantum info review

In quantum information processing we encounter entities such as



where $\hat{U}_F |x_1 x_0\rangle|y\rangle = |x_1 x_0\rangle|y \oplus f(x_1 x_0)\rangle$.

What does this mean? Each line refers to a distinct physical system – in this case a two-state system – or a qubit. So we have

abstract rep

qubit 3 ——

qubit 2 ——

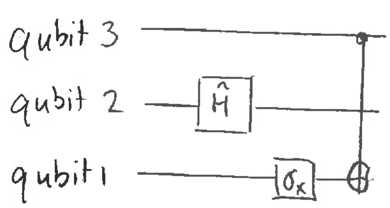
qubit 1 ——

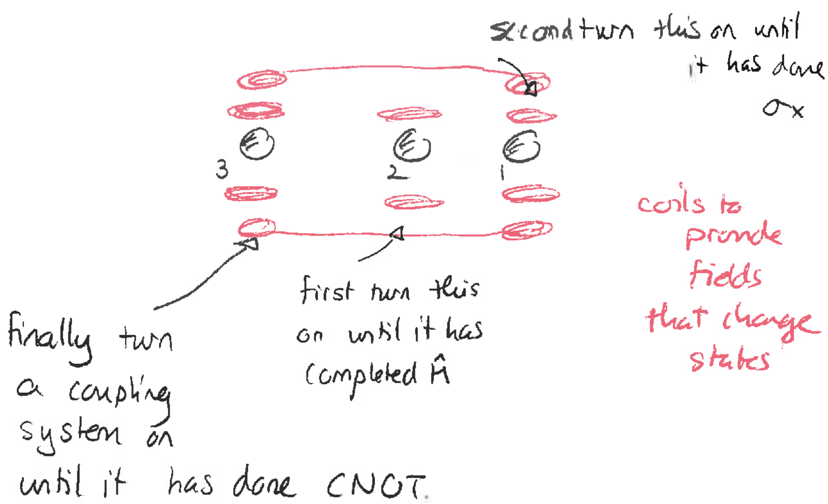
physical situation

The state of a qubit is described by a ket $|\psi\rangle$ with some label. There is a distinction between state and qubit:

- qubit – actual physical object
- state – mathematical object (ket) that is used to determine measurement outcomes and their probabilities and can be given a physical interpretation in these terms

The series of gates is meant to be read left to right and indicates a sequence of operations (evolutions) to which the qubits are subjected





<u>States</u> 1) For a single qubit, the states are represented via kets of the form

$$|0\rangle, |1\rangle, \quad \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle), \quad \frac{1}{\sqrt{2}}(|0\rangle+i|1\rangle)$$

and these correspond to particular physical states. All such states are equally meaningful physically, we really just represent states of the form $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ in terms of $|0\rangle, |1\rangle$ just to facilitate calculations. So for spin-½ these are

$$|+\hat{z}\rangle, |-\hat{z}\rangle, |+\hat{x}\rangle, |+\hat{y}\rangle$$
$$|+\hat{y}\rangle = \frac{1}{\sqrt{2}}(|+\hat{z}\rangle+i|-\hat{z}\rangle)$$

are all equally valid + equally important. We just use $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$ as a basis for convenience of calculations.

2) For multiple qubits, states are represented via tensor products and sums of these: e.g.

$$|\psi\rangle|\varphi\rangle|\chi\rangle = |\psi\rangle\otimes|\varphi\rangle\otimes|\chi\rangle$$

qubit 3 qubit 2 qubit 1

and superpositions of these are allowed e.g.

$$\frac{1}{\sqrt{2}}(|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle)$$

Qubit states are often abbreviated using e.g.

$$|000\rangle \equiv |0\rangle_3|0\rangle_2|0\rangle_1 = |0\rangle_3\otimes|0\rangle_2\otimes|0\rangle_1$$

qubit 3 qubit 2 qubit 1

Some of these states can be factorized into a product state. Others can not

1 Qubit states

For each of the following states describe, the number of qubits for which this is a state. If the state represents multiple qubits indicate which terms in the state pertain to which qubit.

a) $|00\rangle$ 2

b) $|10\rangle$ 2

c) $|11\rangle$ 2

d) $|001\rangle$ 3

e) $|011\rangle$ 3

f) $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ 2

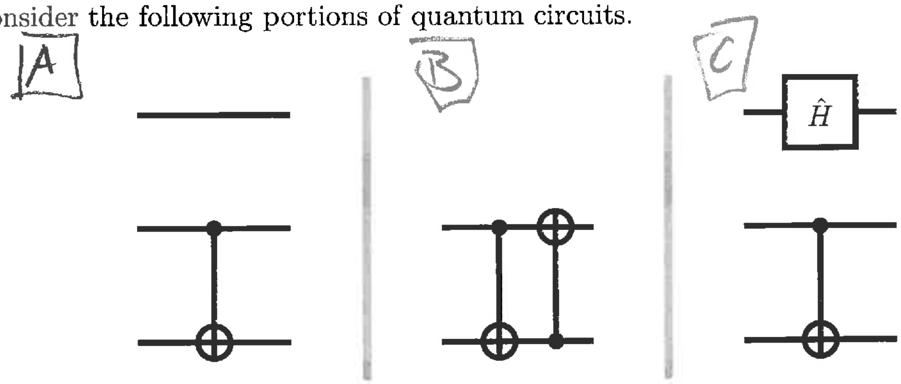
g) $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle - |11\rangle)$ 2

h) $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$ 2

i) $\frac{1}{\sqrt{2}}(|010\rangle - |101\rangle)$ 3

2 Qubit states and circuits

Consider the following portions of quantum circuits.



Now consider the following possible states. Identify which of these could serve as inputs for which circuit (multiple circuits may be possible for each). Identify which terms in the state pertain to which qubit.

a) $|11\rangle$ $\boxed{B}$

b) $|101\rangle$ $\boxed{A}$ $\boxed{C}$

c) $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ $\boxed{B}$

d) $\frac{1}{\sqrt{2}}(|011\rangle + |110\rangle)$ $\boxed{A}$ $\boxed{C}$

e) $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle - |11\rangle)$ $\boxed{B}$

Operations Gates and operations are linear evolutions of a state. For a generic quantum system

$$|\psi_i\rangle \rightarrow |\psi_f\rangle = \hat{U}\,|\psi_i\rangle$$

where $\hat{U}$ is unitary. The effect of an operation can be determined by describing

i) the matrix for the operation

ii) the action of the operation on each basis state

* For a single qubit we need to know

$$\hat{U}|0\rangle \text{ and } \hat{U}|1\rangle$$

to describe the action on any state

$$|\psi_i\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

$$\downarrow \hat{U}$$

$$|\psi_f\rangle = \alpha_0\,\hat{U}\cdot|0\rangle + \alpha_1\hat{U}\,|1\rangle$$

* For multiple qubits we need to describe the action of the operator on all basis states. For example on three qubits we need

$$\hat{U}\,|000\rangle\,;\ \hat{U}\,|001\rangle,\ \hat{U}\,|010\rangle, \ldots$$

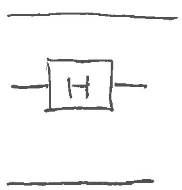
* A tensor product of operators acts as:

$$\hat{U}_2 \otimes \hat{U}_1\ |00\rangle = (\hat{U}_2|0\rangle)(\hat{U}_1|0\rangle)$$

$$\hat{U}_2 \otimes \hat{U}_1\ |01\rangle = (\hat{U}_2|0\rangle)(\hat{U}_1|1\rangle)$$

etc, ...

* when multiple qubits are present, a single qubit gate means that one should act on the qubit element



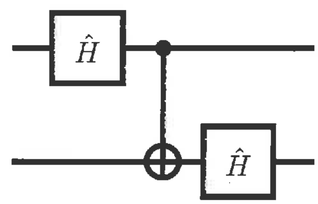
maps $\alpha_0|000\rangle + \alpha_1|001\rangle + \alpha_2|010\rangle + \cdots$

$$\to \alpha_0|0\rangle(\hat{H}|0\rangle)|0\rangle + \alpha_1|0\rangle(\hat{H}|0\rangle)|1\rangle + \cdots \quad \text{etc.}$$

This generalizes to two qubit operations on two of many qubits

3 Operations on multiple qubits

Consider the following quantum circuit.



The initial state is $|01\rangle$.

a) Determine the state of the system after the leftmost Hadamard.

b) Determine the state of the system after the CNOT.

c) Determine the state of the system after the rightmost Hadamard.

d) Suppose that a computational basis measurement is done on both qubits after the rightmost Hadamard. List the probabilities with which the various outcomes occur.

Answer:

a) $$|01\rangle \rightarrow (\hat{H}|0\rangle)|1\rangle$$
$$= \tfrac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|1\rangle$$
$$= \tfrac{1}{\sqrt{2}}(|01\rangle+|11\rangle)$$

b) $$\tfrac{1}{\sqrt{2}}(|01\rangle+|11\rangle) \rightarrow \tfrac{1}{\sqrt{2}}(|01\rangle+|10\rangle)$$

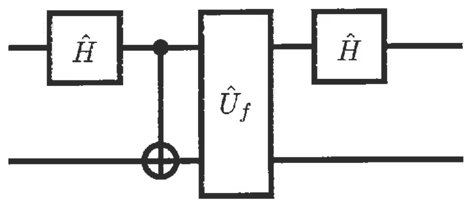
c) $$\tfrac{1}{\sqrt{2}}(|01\rangle+|10\rangle) = \tfrac{1}{\sqrt{2}}|0\rangle|1\rangle + \tfrac{1}{\sqrt{2}}|1\rangle|0\rangle$$
$$\rightarrow \tfrac{1}{\sqrt{2}}|0\rangle(\hat{H}|1\rangle) + \tfrac{1}{\sqrt{2}}|1\rangle\hat{H}|0\rangle$$
$$= \tfrac{1}{\sqrt{2}}|0\rangle\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right) + \tfrac{1}{\sqrt{2}}|1\rangle\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)$$
$$= \tfrac{1}{2}\left[|00\rangle - |01\rangle + |10\rangle + |11\rangle\right]$$

d)

Upper	lower	Prob	
0	0	1/4	$= \lvert\langle 00\lvert\psi_F\rangle\rvert^2$
0	1	1/4	$= \lvert\langle 01\lvert\psi_F\rangle\rvert^2$
1	0	1/4	
0	0	1/4	

4 Oracle invocation

Consider the following quantum circuit, where $\hat{U}_f$ is the standard oracle for a function that maps one qubit to one qubit.



The initial state is $|00\rangle$.

a) Determine an expression for the state of the system after final Hadamard.

b) Evaluate the expression for $f(x) = 0$. What could a computational basis measurement on the upper qubit give?

c) Evaluate the expression for $f(x) = 1$. What could a computational basis measurement on the upper qubit give?

d) Evaluate the expression for $f(x) = x$. What could a computational basis measurement on the upper qubit give?

e) Evaluate the expression for $f(x) = 1 \oplus x$. What could a computational basis measurement on the upper qubit give?

Answer: a)

$$|00\rangle \xrightarrow{\hat{H}\otimes\hat{I}} (H|0\rangle)|0\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle \quad = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$\xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\xrightarrow{\hat{U}_f} \frac{1}{\sqrt{2}}\left\{|0\rangle|0\oplus f(0)\rangle + |1\rangle|1\oplus f(1)\rangle\right\}$$

$$\xrightarrow{\hat{H}\otimes\hat{I}} \frac{1}{\sqrt{2}}\left\{\hat{H}|0\rangle|0\oplus f(0)\rangle + \hat{H}|1\rangle|1\oplus f(1)\rangle\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|0\oplus f(0)\rangle + \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)|1\oplus f(1)\rangle\right\}$$

$$|\psi_f\rangle = \frac{1}{2}\left\{|0\rangle|0\oplus f(0)\rangle + |1\rangle|0\oplus f(0)\rangle + |0\rangle|1\oplus f(1)\rangle - |1\rangle|1\oplus f(1)\rangle\right\}$$

b) $|\psi_f\rangle = \frac{1}{2}\left\{|0\rangle|0\rangle + |1\rangle|0\rangle + |0\rangle|1\rangle - |1\rangle|1\rangle\right\}$

measurement could yield 0 or 1 with prob 1/2

c) $|\psi_f\rangle = \frac{1}{2}\left\{|0\rangle|1\rangle + |1\rangle|1\rangle + |0\rangle|0\rangle - |1\rangle|0\rangle\right\}$ 0 or 1 with prob 1/2

d) $|\psi_f\rangle = \frac{1}{2}\left\{|0\rangle|0\rangle + |1\rangle|0\rangle + |0\rangle|0\rangle - |1\rangle|0\rangle\right\} = |0\rangle|0\rangle$

e) $|\psi_f\rangle = \frac{1}{2}\left\{|0\rangle|1\rangle + |1\rangle|1\rangle + |0\rangle|1\rangle - |1\rangle|1\rangle\right\} = |0\rangle|1\rangle$

$\Big\}$ yields 0. with prob 1