

Tues: HW by 5pm

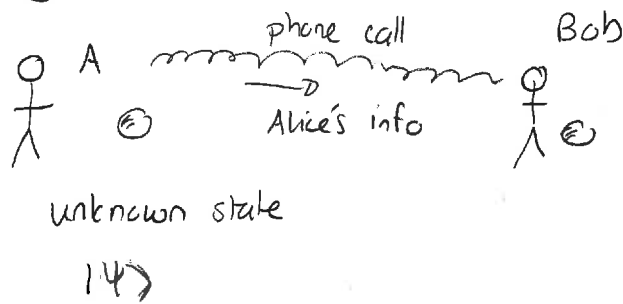
Have all received list of possible projects?

Thurs: R 6.1, 6.2.1 ~~2~~

B 3.7, 3.8

Teleportation

Suppose that Alice has a single qubit in any unknown state. She has a collaborator, Bob, who would like to know the state of her qubit. However, she is not allowed to send the qubit to Bob - she can only send conventional classical information. To facilitate the process



process exists



Bob has at least one qubit and the state of Alice's qubit must appear in Bob's qubit. This must work for any qubit state  $|\psi\rangle$ .

Consider a general initial state for Alice

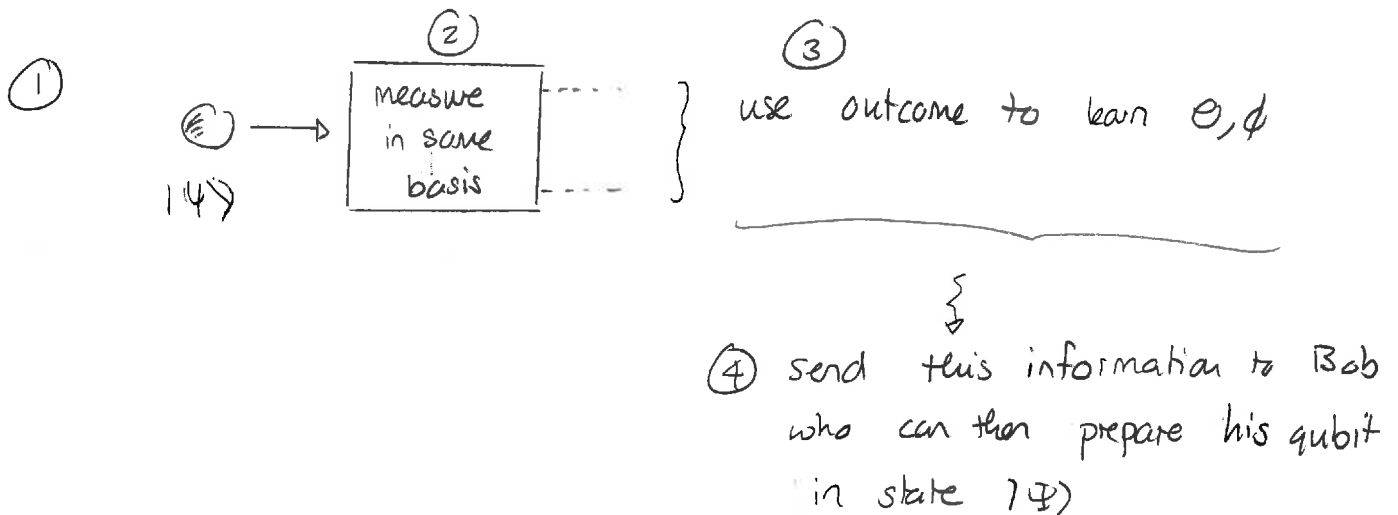
$$|\Psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

where  $\alpha_0$  and  $\alpha_1$  are complex numbers that satisfy  $|\alpha_0|^2 + |\alpha_1|^2 = 1$ .

One strategy for Alice would be to perform a measurement on her qubit, learn  $\alpha_0, \alpha_1$  and transmit this information to Bob. Is this possible? Consider the Bloch sphere representation of the qubit state

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

So Alice might try:



## 1 Information extraction via measurement

Suppose that a qubit is in an unknown state

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle.$$

The following exercises illustrate how much can be learned about the two parameters,  $\theta$  and  $\phi$ , from a measurement on a single copy of this qubit.

- a) Suppose that the qubit is measured in the basis  $\{|0\rangle, |1\rangle\}$ . Determine the probabilities with which the two outcomes occur. If the outcome were that corresponding to the state  $|0\rangle$ , what does this imply about  $\theta$  and  $\phi$  prior to measurement? If the outcome were that corresponding to the state  $|1\rangle$ , what does this imply about  $\theta$  and  $\phi$  prior to measurement?
- b) Suppose that the qubit is measured in the basis

$$\left\{ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right\}$$

Determine the probabilities with which the two outcomes occur. If the outcome were that corresponding to the state  $(|0\rangle + |1\rangle)/\sqrt{2}$ , what does this imply about  $\theta$  and  $\phi$  prior to measurement? If the outcome were that corresponding to the state  $(|0\rangle - |1\rangle)/\sqrt{2}$ , what does this imply about  $\theta$  and  $\phi$  prior to measurement?

Answer:

$$\begin{aligned} \text{a) Prob}(0) &= |\langle 0|\Psi\rangle|^2 = \cos^2(\theta/2) \\ \text{Prob}(1) &= |\langle 1|\Psi\rangle|^2 = \sin^2(\theta/2) \end{aligned}$$

If outcome = 0 then  $\cos^2(\theta/2) \neq 0$  so  $\theta \neq \pi$

If " = 1 "  $\sin^2(\theta/2) \neq 0$  so  $\theta \neq 0$



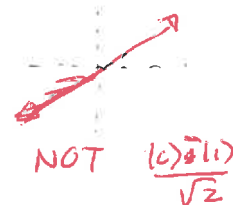
$$\begin{aligned} \text{b) Prob}(+) &= \left| \langle \frac{|0\rangle + |1\rangle}{\sqrt{2}} | \Psi \rangle \right|^2 = \left( \frac{1}{\sqrt{2}} \right)^2 \left| \cos\theta/2 + e^{i\phi}\sin\theta/2 \right|^2 \\ &= \frac{1}{2} (\cos\theta/2 + e^{i\phi}\sin\theta/2)(\cos\theta/2 + e^{-i\phi}\sin\theta/2) \\ &= \frac{1}{2} (\cos^2\theta/2 + \sin^2\theta/2 + \sin\theta/2\cos\theta/2(e^{i\phi} + e^{-i\phi})) \\ &= \frac{1}{2} (1 + \sin\theta/2\cos\theta/2 \cdot 2\cos\phi) \end{aligned}$$

So Prob (+) =  $\frac{1}{2}(1 + \sin\theta \cos\phi)$

similarly

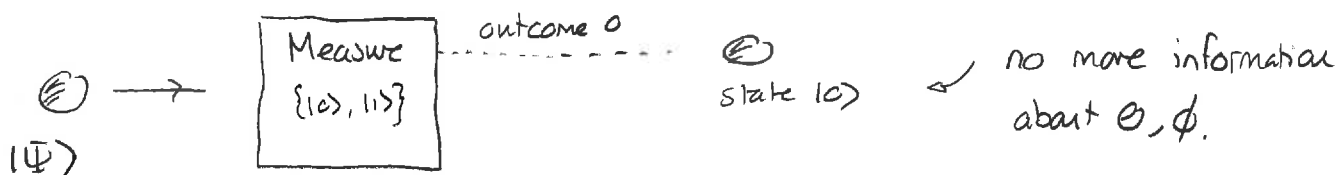
Prob (-) =  $\frac{1}{2}(1 - \sin\theta \cos\phi)$

If outcome = + then  $\sin\theta \cos\phi \neq -1 \Rightarrow$   
 $\Leftrightarrow \theta \neq \pi/2$  and  $\phi \neq \pi$



If outcome = - then  $\theta \neq \pi/2$  and  $\phi \neq 0$   $\square$

So these measurements only yield very limited information and  $\theta, \phi$ . We could show that the situation is comparable for any arbitrary single qubit measurement. Additionally repeating the measurement is not useful. For example

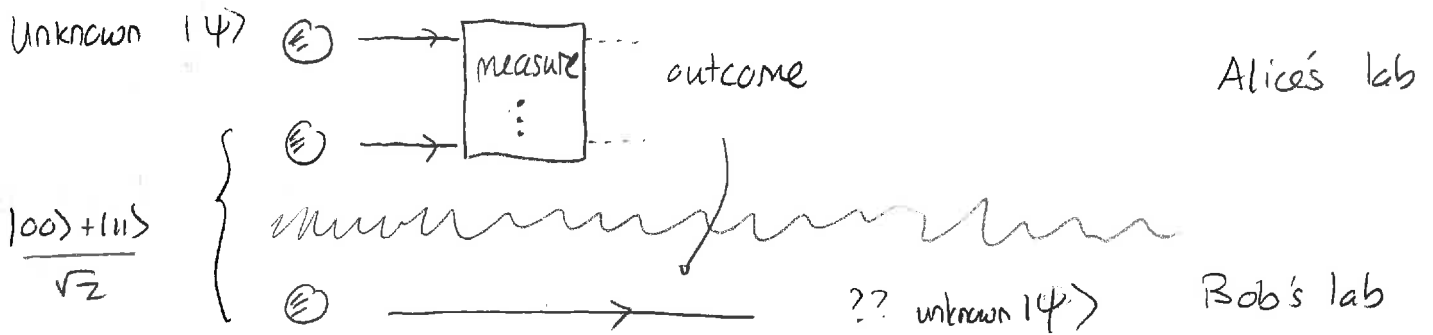


With multiple copies, we could potentially assemble statistics to learn about  $\theta, \phi$ . But this is not the situation.

Separately the no-cloning theorem prohibits copying a state - if so this would facilitate learning  $\theta, \phi$ .



② Alice does operations between her two qubits



③ Alice sends results of her measurements to Bob who can reconstruct the state  $|\psi\rangle$ .

We now consider the mathematics behind this.

## 2 Quantum teleportation part I

Suppose that Alice's qubit is in an unknown state

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle.$$

and that Alice and Bob share a separate entangled pair of qubits in the state

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle).$$

The state of all three qubits is

$$|\psi\rangle |\Phi_+\rangle.$$

a) Expand  $|\psi\rangle |\Phi_+\rangle$  in terms of  $|00\rangle |0\rangle, |00\rangle |1\rangle, |01\rangle |0\rangle, \dots$

Recall Bell states

$$|\Phi_+\rangle := \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

$$|\Phi_-\rangle := \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)$$

$$|\Psi_+\rangle := \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)$$

$$|\Psi_-\rangle := \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle).$$

Thus

$$|00\rangle := \frac{1}{\sqrt{2}} (|\Phi_+\rangle + |\Phi_-\rangle)$$

$$|11\rangle := \frac{1}{\sqrt{2}} (|\Phi_+\rangle - |\Phi_-\rangle)$$

$$|01\rangle := \frac{1}{\sqrt{2}} (|\Psi_+\rangle + |\Psi_-\rangle)$$

$$|10\rangle := \frac{1}{\sqrt{2}} (|\Psi_+\rangle - |\Psi_-\rangle).$$

b) Rewrite  $|\psi\rangle |\Phi_+\rangle$  in the form

$$|\psi\rangle |\Phi_+\rangle = |\Phi_+\rangle |\varphi_0\rangle + |\Phi_-\rangle |\varphi_1\rangle + |\Psi_-\rangle |\varphi_2\rangle + |\Psi_+\rangle |\varphi_3\rangle.$$

Answer a)

$$\begin{aligned}
 |\Psi\rangle|\Phi_+\rangle &= (\alpha_0|0\rangle + \alpha_1|1\rangle) \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \\
 &= \frac{\alpha_0}{\sqrt{2}} |000\rangle + \frac{\alpha_0}{\sqrt{2}} |011\rangle + \frac{\alpha_1}{\sqrt{2}} |100\rangle + \frac{\alpha_1}{\sqrt{2}} |111\rangle \\
 &= \frac{\alpha_0}{\sqrt{2}} |00\rangle|0\rangle + \frac{\alpha_0}{\sqrt{2}} |01\rangle|1\rangle + \frac{\alpha_1}{\sqrt{2}} |10\rangle|0\rangle + \frac{\alpha_1}{\sqrt{2}} |11\rangle|1\rangle
 \end{aligned}$$

b)

$$\begin{aligned}
 |\Psi\rangle|\Phi_+\rangle &= \frac{\alpha_0}{2} [|\Phi_+\rangle|0\rangle + |\Phi_-\rangle|0\rangle] \\
 &\quad + \frac{\alpha_0}{2} [|\Psi_+\rangle|1\rangle + |\Psi_-\rangle|1\rangle] \\
 &\quad + \frac{\alpha_1}{2} [|\Phi_+\rangle|0\rangle - |\Phi_-\rangle|0\rangle] \\
 &\quad + \frac{\alpha_1}{2} [|\Phi_+\rangle|1\rangle - |\Phi_-\rangle|1\rangle]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} |\Phi_+\rangle \overbrace{(\alpha_0|0\rangle + \alpha_1|1\rangle)}^{|\Phi_0\rangle} \\
 &\quad + \frac{1}{2} |\Phi_-\rangle (\alpha_0|0\rangle - \alpha_1|1\rangle) \\
 &\quad + \frac{1}{2} |\Psi_+\rangle (\alpha_0|1\rangle + \alpha_1|0\rangle) \\
 &\quad + \frac{1}{2} |\Psi_-\rangle (\alpha_0|1\rangle - \alpha_1|0\rangle)
 \end{aligned}$$



Alice

Bob



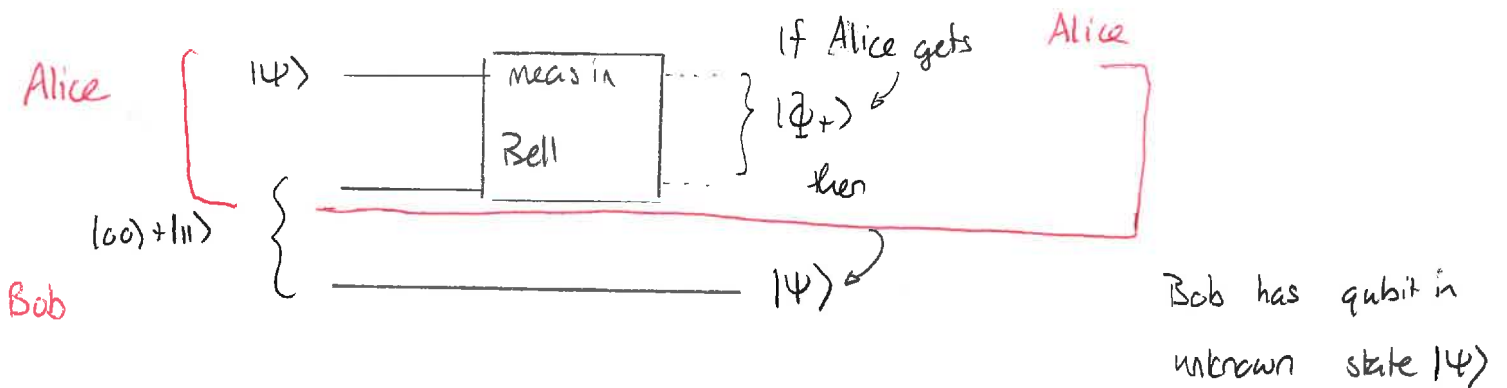
Now the Bell states form an orthonormal set. Thus it is possible to measure a pair of qubits in the Bell basis.

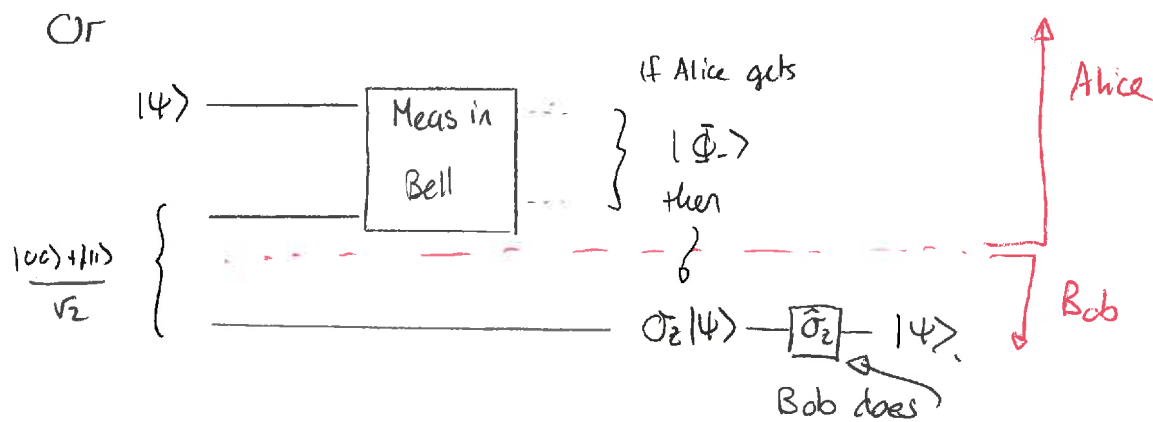
Suppose that Alice measures her pair in the Bell basis:

We tabulate the results:

Associated state for Alice	Prob	Bob state
$ \Phi_+\rangle$	$\frac{1}{4}$	$\alpha_0 0\rangle + \alpha_1 1\rangle =  \psi\rangle$
$ \Phi_-\rangle$	$\frac{1}{4}$	$\alpha_0 0\rangle - \alpha_1 1\rangle = \hat{\sigma}_z  \psi\rangle$
$ \Psi_+\rangle$	$\frac{1}{4}$	$\alpha_0 1\rangle + \alpha_1 0\rangle = \hat{\sigma}_x  \psi\rangle$
$ \Psi_-\rangle$	$\frac{1}{4}$	$\alpha_0 1\rangle - \alpha_1 0\rangle = \hat{\sigma}_x \hat{\sigma}_z  \psi\rangle$

So all that Alice needs to do is measure in the Bell basis and describe which of the associated states she obtained and send that information to Bob. He can then reverse the modification to extract  $|\psi\rangle$ . So





For each possible Bell basis outcome Bob can apply a corrective unitary and regain  $|\psi\rangle$  regardless of  $|\psi\rangle$ . All Alice must do is communicate the result of her measurement.