Tues: HW by 5pm

Thurs: R 5.2.4 B 3.4 – 3.5

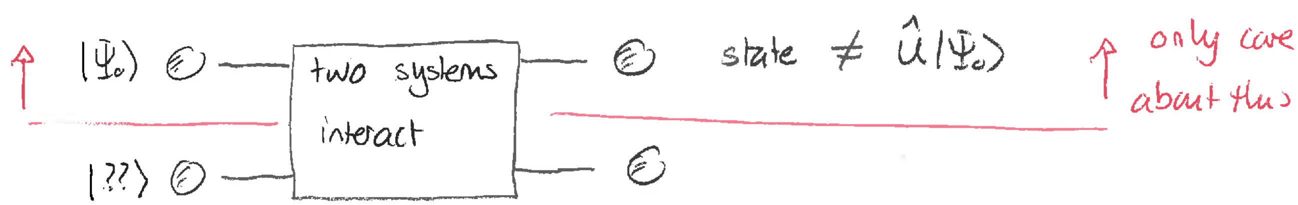
R 5.4

Quantum evolution

The general rule for evolution of a closed quantum system (does not become entangled with external systems) is

> If the initial state of the system prior to evolution is $|\Psi_0\rangle$ then the state after evolution is
>
> $$|\Psi\rangle = \hat{U}\,|\Psi_0\rangle$$
>
> where $\hat{U}$ is a unitary linear operator, i.e.
>
> $$\hat{U}^{+}\hat{U} = \hat{I}$$

This rule applies generally to closed quantum systems. It has to be modified if the system in consideration interacts with an external quantum system



Single qubit operations

The description of single qubit operations is simplified by using the Pauli operators

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we can show that these satisfy

i) $\hat{\sigma}_m^{+} = \hat{\sigma}_m$

ii) $\hat{\sigma}_m \hat{\sigma}_n = \hat{m}\cdot\hat{n}\ \hat{I} + i\left[(\hat{m}\times\hat{n})_x \hat{\sigma}_x + (\hat{m}\times\hat{n})_y \hat{\sigma}_y + (\hat{m}\times\hat{n})_z \hat{\sigma}_z\right]$

for $\hat{m}, \hat{n}$ in $\{\hat{x}, \hat{y}, \hat{z}\}$. These will ultimately reduce matrix operations to algebra operations.

<u>Example</u>: Use the above to show that $\hat{\sigma}_m$ are unitary

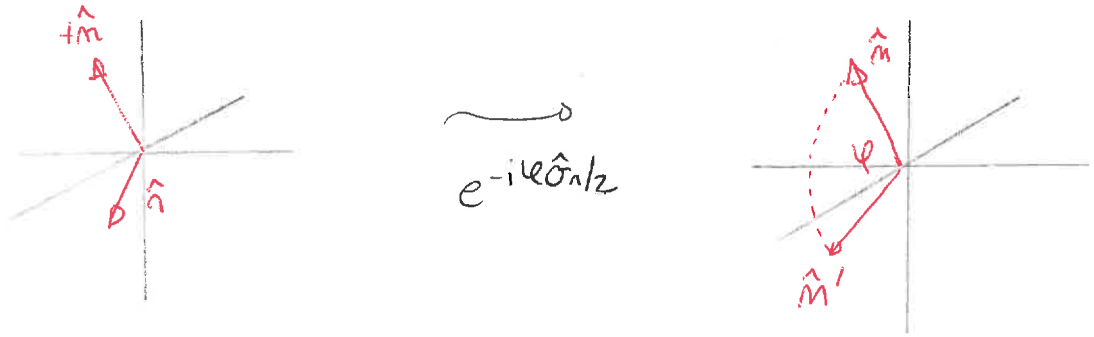
<u>Answer</u>: $\hat{\sigma}_m^{+} \hat{\sigma}_m = \hat{\sigma}_m \hat{\sigma}_m$

$= \hat{m}\cdot\hat{m}\ \hat{I} + 0 = \hat{I}$ ∎

We can use these and the definition of exponentiation to show that:

> For any unit vector $\hat{n} = n_x \hat{x} + n_y \hat{y} + n_z \hat{z}$
>
> $e^{-i\varphi \hat{\sigma}_n / 2}$
>
> where $\hat{\sigma}_n = n_x \hat{\sigma}_x + n_y \hat{\sigma}_y + n_z \hat{\sigma}_z$ is unitary

We can also show that this is represented as a rotation in the Bloch sphere

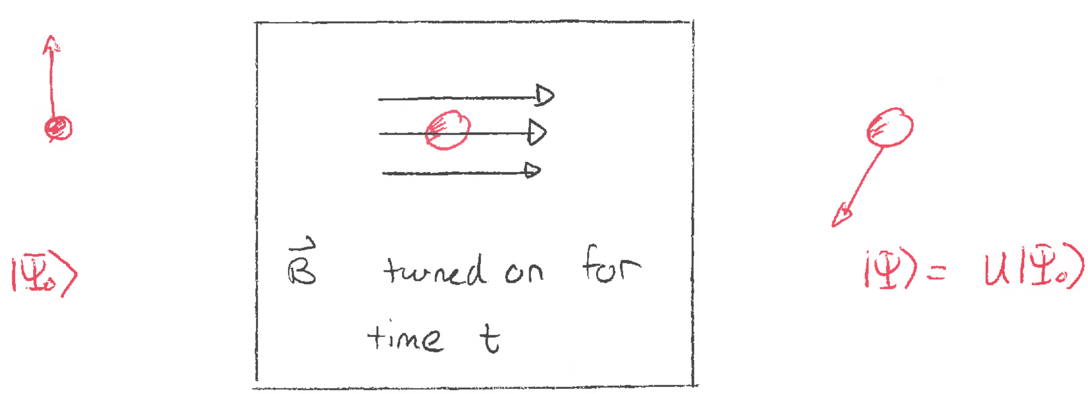


$$|\Psi_0\rangle \approx |+\hat{m}\rangle \qquad\qquad |\Psi\rangle = \xi e^{-i\varphi\hat{\sigma}\cdot\hat{n}/2}|\hat{m}\rangle \cong |\hat{m}'\rangle$$

Finally an algebraic operation shows that any single qubit unitary operation can be represented (up to a global phase) by such a rotation.

Constructing single qubit rotations

We generally construct single qubit rotations by allowing the qubit to evolve for a duration of time under a specific external influence. For example we may subject a spin-½ particle to a magnetic field that is turned on for a duration of t



We consider how to construct such an evolution.

1 Time varying evolution

Consider the single qubit unitary operator

$$\hat{U}(t) := e^{-i\omega t\hat{\sigma}_z/2}.$$

Suppose that the state of the qubit at $t = 0$ is $|\Psi_0\rangle$. The at time t its state will be

$$|\Psi(t)\rangle = \hat{U}(t)\,|\Psi_0\rangle$$

a) Show that

$$i\hbar\frac{\partial}{\partial t}\,|\Psi(t)\rangle = \hat{H}\,|\Psi(t)\rangle$$

where $\hat{H}$ is a Hermitian operator that is independent of t. Determine an expression for $\hat{H}$.

The operator $\hat{H}$ is the Hamiltonian for the system and is an operator which represents the energy of the system and is usually constructed from the energy for the analogous classical system. For a classical magnetic dipole the energy in an external magnetic field is $U = -\boldsymbol{\mu}\cdot\mathbf{B}$. For a classical system

$$\boldsymbol{\mu} = \frac{gq}{2m}\mathbf{S}$$

where $\mathbf{S}$ is the spin angular momentum of the system. The translation into a quantum operator involves the identification:

$$S_x \leftrightarrow \frac{\hbar}{2}\hat{\sigma}_x$$
$$S_y \leftrightarrow \frac{\hbar}{2}\hat{\sigma}_y$$
$$S_z \leftrightarrow \frac{\hbar}{2}\hat{\sigma}_z$$

b) Starting with the classical energy write an expression for the general possible $\hat{H}$ in terms of the Pauli operators and the external magnetic field components.

c) Determine the field direction that results in the unitary evolution of this problem.

d) Determine an expression for ω in terms of relevant parameters in this problem.

Answer: a) $|\Psi(t)\rangle = \hat{U}(t)\,|\Psi_0\rangle$

$$\Rightarrow \frac{\partial}{\partial t}|\Psi(t)\rangle = \frac{\partial \hat{U}(t)}{\partial t}\,|\Psi_0\rangle$$

$$= \frac{\partial}{\partial t} e^{-i\omega t\hat{\sigma}_z/2}\quad|\Psi_0\rangle$$

$$\Rightarrow \frac{\partial}{\partial t}|\Psi(t)\rangle = -\frac{i\omega\hat{\sigma}_z}{2} e^{-i\omega t\hat{\sigma}_z/2}|\Psi_0\rangle$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \frac{\hbar\omega}{2}\hat{\sigma}_z \underbrace{e^{-i\omega t\hat{\sigma}_z/2}|\Psi_0\rangle}_{|\Psi(t)\rangle}$$

So it works with $\hat{H} = \frac{\hbar\omega}{2}\hat{\sigma}_z$

b) $U = -\vec{\mu}\cdot\vec{B} = -[\mu_x B_x + \mu_y B_y + \mu_z B_z]$

$$= -\frac{gq}{2m}(S_x B_x + S_y B_y + S_z B_z)$$

$$\longrightarrow -\frac{gq}{2m}\frac{\hbar}{2}(\hat{\sigma}_x B_x + \hat{\sigma}_y B_y + \hat{\sigma}_z B_z)$$

c) We need $B_x, B_y = 0$. So field is along $\hat{z}$

d) $\frac{\cancel{\hbar}\omega}{\cancel{2}} = -\frac{gq}{2m}B_z\frac{\cancel{\hbar}}{\cancel{2}} \Rightarrow \omega = -\frac{gq}{2m}B_z$

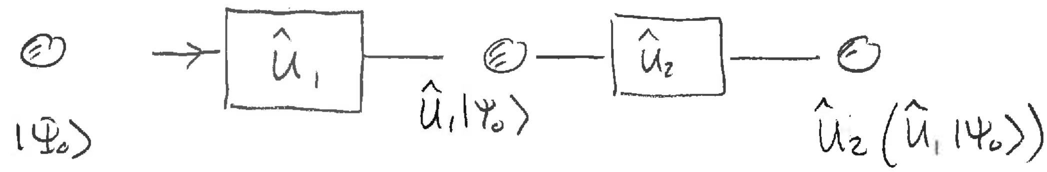
So we can generate this unitary via

1 No field applied $\longrightarrow$ 2 Apply field along $\hat{z}$ with $B = -\frac{2m}{gq}\omega$ for time t $\longrightarrow$ 3 Turn field off

$|\Psi_0\rangle \longrightarrow \hat{U} = e^{-i\omega t\hat{\sigma}_z/2} \longrightarrow |\Psi(t)\rangle = \hat{U}(t)|\Psi_0\rangle$

Composition of gates.

In general we expect that two successive evolutions constitute a single evolution



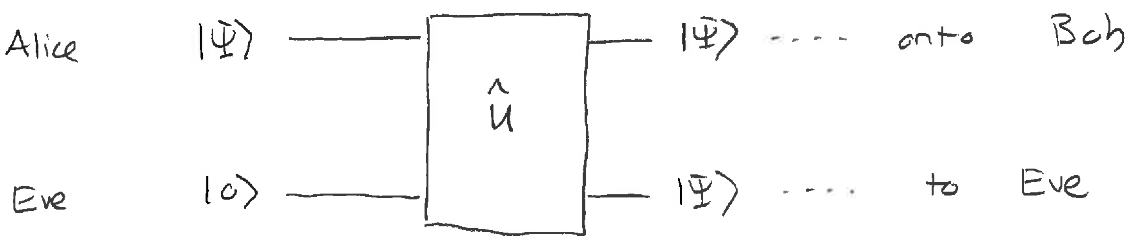
So is $\hat{U}_2\hat{U}_1$ an allowed evolution. It will definitely be linear. Then let $\hat{U} = \hat{U}_2\hat{U}_1$

$$
\begin{aligned}
\hat{U}^{+}\hat{U} &= (\hat{U}_2\hat{U}_1)^{+}\hat{U}_2\hat{U}_1 \\
&= \hat{U}_1^{+}\underbrace{\hat{U}_2^{+}\hat{U}_2}_{\hat{I}}\hat{U}_1 \\
&= \hat{U}_1^{+}\hat{U}_1 \\
&= \hat{I}
\end{aligned}
$$

So the product does represent an evolution.

Two qubit operations

The same general rules apply for two qubit operations and we can now see if they allow Eve to produce a cloning device. So we need



to work for all states $|\Psi\rangle = |0\rangle, |1\rangle, \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

Such a device must clearly map

$$\underbrace{|0\rangle}_{\text{Alice}}\underbrace{|0\rangle}_{\text{Eve}} \xrightarrow{\hat{U}} \underbrace{|0\rangle}_{A}\underbrace{|0\rangle}_{E}$$

$$|1\rangle|0\rangle \xrightarrow{\hat{U}} |1\rangle|1\rangle.$$

Then by linearity this maps

$$\frac{1}{\sqrt{2}}\underbrace{(|0\rangle \pm |1\rangle)}_{\text{Alice}}\underbrace{|0\rangle}_{\text{Eve}} = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|0\rangle) \xrightarrow[\text{linearity}]{\hat{U}} \frac{1}{\sqrt{2}}\hat{U}|0\rangle|0\rangle \pm \frac{1}{\sqrt{2}}\hat{U}|1\rangle|0\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle|0\rangle \pm \frac{1}{\sqrt{2}}|1\rangle|1\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle)$$

But we had desire the output $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

$$= \frac{1}{2}(|0\rangle|0\rangle \pm |0\rangle|1\rangle \pm |1\rangle|0\rangle + |1\rangle|1\rangle)$$

and these states are clearly not the same. Linearity thus prohibits this type of cloning device. This is an example of the no-cloning theorem

Two qubit gates

In general a two qubit gate will map all possible basis states of the pair of qubits: So

$$\hat{U}|00\rangle = U_{00}|00\rangle + U_{10}|01\rangle + U_{20}|10\rangle + U_{30}|11\rangle$$

$$\hat{U}|01\rangle = U_{01}|00\rangle + U_{11}|11\rangle + U_{21}|10\rangle + U_{31}|11\rangle$$

$$\vdots$$

where U_{ij} are complex quantities that can be represented in a matrix:

$$\hat{U} \sim \begin{pmatrix} U_{00} & U_{01} & U_{02} & U_{03} \\ U_{10} & U_{11} & U_{12} & U_{13} \\ U_{20} & U_{21} & U_{22} & U_{23} \\ U_{30} & U_{31} & U_{32} & U_{33} \end{pmatrix}$$

Many such two qubit gates can be represented conveniently via tensor products of single qubit gates or via sums of tensor products. In order to do algebra with these we note the following tensor product results

1) $(\hat{A}\otimes\hat{B}_1)\cdot(\hat{A}_2\otimes\hat{B}_2) = \hat{A}_1\hat{A}_2 \otimes \hat{B}_1\hat{B}_2$

2) $(\hat{A}\otimes\hat{B})^{+} = \hat{A}^{+}\otimes\hat{B}^{+}$

3) $(\hat{A}_1\otimes\hat{B}_1 + \hat{A}_2\otimes\hat{B}_2)^{+} = (A_1\otimes B_1)^{+} + (A_2\otimes B_2)^{+}$

etc...

2 Tensor products of gates

Consider

$$\hat{U}_1 = \hat{\sigma}_z \otimes \hat{\sigma}_x$$
$$\hat{U}_2 = \hat{\sigma}_z \otimes \hat{\sigma}_y.$$

a) Express each of these in terms of matrices.

b) Show that each of these is unitary.

c) Determine a matrix expression for $\hat{U} = \hat{U}_1\hat{U}_2$ and show that it matches the expression obtained via tensor product algebra.

Answer: a)

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\hat{U}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

b) $$\hat{U}_1^+ \hat{U}_1 = (\hat{\sigma}_z^+ \otimes \hat{\sigma}_x^+)(\hat{\sigma}_z \otimes \hat{\sigma}_x) = \underbrace{\sigma_z^+\sigma_z}_{\hat{I}} \otimes \underbrace{\hat{\sigma}_x^+ \hat{\sigma}_x}_{\hat{I}} = \underbrace{\hat{I} \otimes \hat{I}}_{\hat{I}}$$

similar for $\hat{U}_2^+\hat{U}_2$

c) $$\hat{U}_1\hat{U}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$= i \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

Tensor alg $\Rightarrow$ $$(\hat{\sigma}_z \otimes \hat{\sigma}_x)(\hat{\sigma}_z \otimes \hat{\sigma}_y) = \underbrace{\sigma_z\sigma_z}_{\hat{I}} \otimes \underbrace{\sigma_x\sigma_y}_{i\hat{\sigma}_z}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$ works