Tues HW by 5pm

Thurs: Seminar / class meeting

Tues:

Measurements in quantum physics

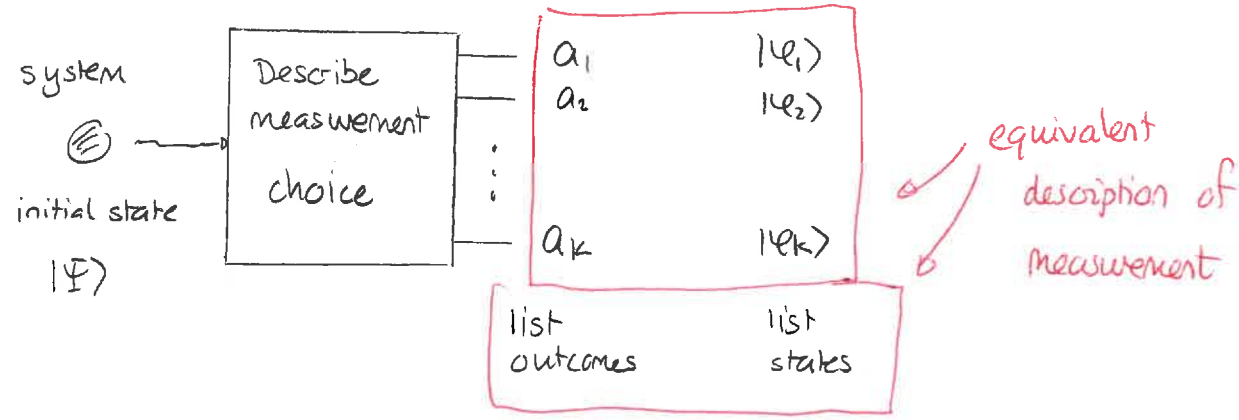
Any measurement scenario in quantum physics has four ingredients:

1) a choice of measurement / type of measurement
2) possible outcomes of the measurement
3) states associated with the outcomes
4) probability of an outcome for a given input state

There is a distinction between all of these that is not always apparent in classical physics. For example, we may say that "we measure the mass of a piece of iron to be 0.200kg." This actually has ingredients:

* choice of measurement = mass (NOT volume, density, position conductivity)
* possible outcomes = any positive real number $0\,kg \rightarrow \infty\, kg$
* associated states = less clear (equal volumes of iron)
* probability = less clear (fluctuations as a result of imperfect measurements)

Schematically in quantum physics a measurement is described via:



Example: Hydrogen atom / electron in H atom

Measurement choice?	Outcomes	States
Energy	E_1	$\vert 100\rangle$
	E_2	$\vert 200\rangle, \vert 211\rangle, \vert 210\rangle, \vert 21{-1}\rangle$ (n l m)
	$\vdots$	$\vdots$
	$E_k = E_1/k^2$	
	$\vdots$	
Magnitude total angular momentum, L^2	0 $(l=0)$	$\vert 100\rangle, \vert 200\rangle, \ldots$
	$2\hbar$ $(l=1)$	$\vert 210\rangle, \vert 211\rangle, \vert 21{-1}\rangle, \ldots$
	$6\hbar$ $(l=2)$	
	$\vdots$	
	$\hbar\, l(l+1)$	$\vert n\,l\,l\rangle, \vert n\,l\,l{-1}\rangle, \ldots$
	$\vdots$	

In all cases the probability rule is the same

If a measurement described via a set of outcomes $\{a_1, a_2, \ldots, a_k\}$ and associated states $\{|\varphi_1\rangle, |\varphi_2\rangle, \ldots |\varphi_k\rangle\}$ is performed on a system in state $|\Psi\rangle$, then

$$\text{Prob (outcome } a_j) = |\langle\varphi_j|\Psi\rangle|^2$$

where $|\varphi_j\rangle$ is the state that gives outcome a_j with certainty

In this scheme the measurement outcomes really just serve as labels. The crucial aspects of describing the measurement are the associated states. For the most "selective" measurement this set of states must form an orthonormal basis for the space of states.

Thus we can begin to describe the measurement by listing an orthonormal basis and then later attach labels. So we say:

"The measurement corresponding to $\{|\varphi_1\rangle, |\varphi_2\rangle, \dots |\varphi_k\rangle\}$."

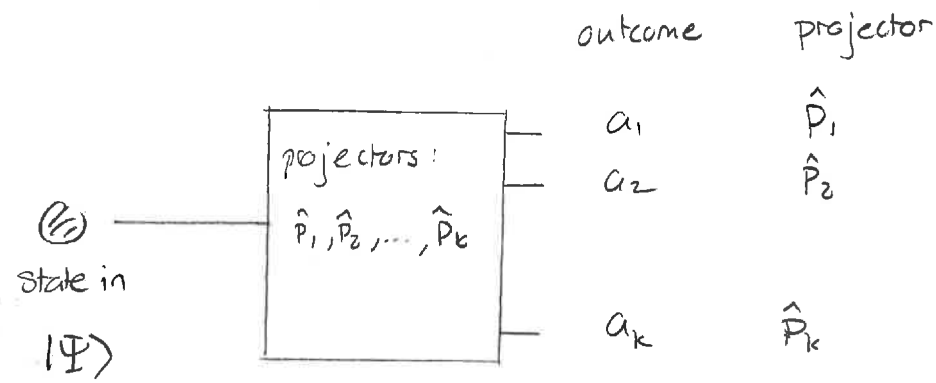
Alternatively we can use these to construct projectors that describe the measurement just as effectively. Here:

> A measurement is described by a set of operators: $\hat{P}_1, \hat{P}_2, \dots, \hat{P}_k$ which satisfy:
>
> 1) $\hat{P}_i$ is positive, i.e. for all $|\Psi\rangle$, $\langle\Psi|\hat{P}_i|\Psi\rangle \geq 0$
> 2) $\sum_{i=1}^{k} \hat{P}_i = \hat{I}$
> 3) $\hat{P}_i^2 = \hat{P}_i$ for all i
> 4) $\hat{P}_i\hat{P}_j = \hat{P}_j\hat{P}_i$

We then need to attach outcomes to each measurement operator. We can do this independently of the above construction.

Given the usual measurement description we can show that $\hat{P}_i = |\varphi_i\rangle\langle\varphi_i|$ satisfy the measurement requirements. The reverse is less straightforward but if the number of outcomes matches the dimension of the ket space, then various linear algebra techniques give an equivalence

In this formalism



Then

$$\text{Prob (outcome } a_i) = \langle \Psi | \hat{P}_i | \Psi \rangle$$

and if outcome a_i is obtained then the state after this is:

$$\frac{\hat{P}_i |\Psi\rangle}{\sqrt{\langle \Psi | \hat{P}_i | \Psi \rangle}}$$

We would not say "measure a_2" for example. Rather we should say "we measure to determine if we get outcome $a_1, a_2, \ldots, a_k$."

Also the choice of measurement is not between a_1 and a_2 or a_1 and $a_3, \ldots$ The choice of measurement is between two entire sets: $\{a_1, a_2, \ldots, a_k\}$ vs $\{b_1, b_2, \ldots, b_k\}$

1 Measurements and states

Consider a spin-1/2 particle. The following exercises use the states

$$
\begin{aligned}
|+\hat{\boldsymbol{x}}\rangle &= \frac{1}{\sqrt{2}}\Big(|+\hat{\boldsymbol{z}}\rangle + |-\hat{\boldsymbol{z}}\rangle\Big)\\
|-\hat{\boldsymbol{x}}\rangle &= \frac{1}{\sqrt{2}}\Big(|+\hat{\boldsymbol{z}}\rangle - |-\hat{\boldsymbol{z}}\rangle\Big)\\
|+\hat{\boldsymbol{y}}\rangle &= \frac{1}{\sqrt{2}}\Big(|+\hat{\boldsymbol{z}}\rangle + i\,|-\hat{\boldsymbol{z}}\rangle\Big)\\
|-\hat{\boldsymbol{y}}\rangle &= \frac{1}{\sqrt{2}}\Big(|+\hat{\boldsymbol{z}}\rangle - i\,|-\hat{\boldsymbol{z}}\rangle\Big)
\end{aligned}
$$

and the standard representation

$$
\begin{aligned}
|+\hat{\boldsymbol{z}}\rangle &\equiv \begin{pmatrix}1\\0\end{pmatrix}\\
|-\hat{\boldsymbol{z}}\rangle &\equiv \begin{pmatrix}0\\1\end{pmatrix}
\end{aligned}
$$

For each of the following choices of measurement, list the measurement outcomes, associated states and associated projector operators.

a) SG $\hat{\boldsymbol{x}}$.

b) SG $\hat{\boldsymbol{y}}$.

Consider photon polarization. The following exercises use the states

$$
\begin{aligned}
|\nearrow\rangle &= \frac{1}{\sqrt{2}}\Big(|\rightarrow\rangle + |\uparrow\rangle\Big)\\
|\nwarrow\rangle &= \frac{1}{\sqrt{2}}\Big(|\rightarrow\rangle - |\uparrow\rangle\Big)\\
|L\rangle &= \frac{1}{\sqrt{2}}\Big(|\rightarrow\rangle + i\,|\uparrow\rangle\Big)\\
|R\rangle &= \frac{1}{\sqrt{2}}\Big(|\rightarrow\rangle - i\,|\uparrow\rangle\Big)
\end{aligned}
$$

and the standard representation

$$
\begin{aligned}
|\rightarrow\rangle &\equiv \begin{pmatrix}1\\0\end{pmatrix}\\
|\uparrow\rangle &\equiv \begin{pmatrix}0\\1\end{pmatrix}
\end{aligned}
$$

For each of the following choices of measurement, list the measurement outcomes, associated states and associated projector operators.

c) Photon polarization along the "diagonal axes."

d) Circular photon polarization.

e) Does the form of the measurement operators for either measurement depend on the measurement outcomes?

f) Would machinery for calculating probabilities use the measurement outcomes?

Answer: a)

Outcome	State	Projector
$S_x = +\hbar/2$	$\|+\hat{x}\rangle$	$\hat{P}_+ = \|+\hat{x}\rangle\langle+\hat{x}\| = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$
$S_x = -\hbar/2$	$\|-\hat{x}\rangle$	$\hat{P}_- = \|-\hat{x}\rangle\langle-x\| = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

$$P_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

$$P_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$$

b)

Outcome	State	Projector
$S_y = +\hbar/2$	$\|+\hat{y}\rangle$	$P_{+i} = \|+\hat{y}\rangle\langle+y\| = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$
$S_y = -\hbar/2$	$\|-\hat{y}\rangle$	$P_{-i} = \|-\hat{y}\rangle\langle-\hat{y}\| = \frac{1}{2}\begin{pmatrix}1 & +i\\ -i & 1\end{pmatrix}$

Here $$|+\hat{y}\rangle\langle+\hat{y}| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$$

$$|-\hat{y}\rangle\langle-\hat{y}| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\begin{pmatrix}1 & +i\end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{2}\begin{pmatrix}1 & +i\\ -i & 1\end{pmatrix}$$

Note that for a,b) we do not choose to measure $+\hbar/2$ (or $-\hbar/2$). We choose to measure S_x or S_y and we will obtain one of the outcomes $+\hbar/2$ or $-\hbar/2$

c)

Outcome	State	Projector
$+45°$	$\lvert\nearrow\rangle$	$\lvert\nearrow\rangle\langle\nearrow\rvert = \hat{P}_{\nearrow} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$
$-45°$	$\lvert\nwarrow\rangle$	$\lvert\nwarrow\rangle\langle\nwarrow\rvert = \hat{P}_{\nwarrow} = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

d)

Outcome	State	Projector
L	$\lvert L\rangle$	$P_L = \lvert L\rangle\langle L\rvert = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$
R	$\lvert R\rangle$	$P_R = \lvert R\rangle\langle R\rvert = \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$

e) Clearly the form of $\hat{P}_{\nearrow}$ is same as $\hat{P}_{+}$, so this does not depend on the type of measurement.

f) No, it just uses the projector + the initial state.

Again note that we do not choose to measure $\nearrow$ or $\nwarrow$. We choose a measurement from a) $\{\nearrow$ vs $\nwarrow\}$ $\leftarrow$ one choice

b) $\{L$ vs $R\}$ $\leftarrow$ one choice

and in each case get one of the two outcomes

The mathematics of calculating the probability only requires the probability operator and the initial state $\lvert\Psi\rangle \sim \begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}$. So to do these we just need projectors and not associated outcomes

Multiple qubit measurements.

Recall that if we have two qubits and measure each in the $\{|0\rangle, |1\rangle\}$ basis then the associated operators are:

outcome A	outcome B	operator
0	0	$\hat{P}_{00} = \|00\rangle\langle 00\|$
0	1	$\hat{P}_{01} = \|01\rangle\langle 01\|$
1	0	$\hat{P}_{10} = \|10\rangle\langle 10\|$
1	1	$\hat{P}_{11} = \|11\rangle\langle 11\|$

and

$$\hat{P}_{00} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \hat{P}_{01} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdots$$

If we only measure on qubit A (left) and want probabilities regardless of outcome of a measurement on B then:

outcome	operator
0	$\hat{P}_{00} + \hat{P}_{01} \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$
1	$\hat{P}_{10} + \hat{P}_{11} \sim \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

We now try to construct these directly from measurement operators for individual qubits.

We start by asking:

Operator $|0\rangle\langle 0|$ for A

Operator $|1\rangle\langle 1|$ for B

$\longrightarrow$ How to get $\hat{P}_{01}$ for A and B

Tensor Products

The construct will involve the tensor product of operators. This is most easily visualized using matrices. Consider a product of states

$$|\Psi\rangle|\Phi\rangle \equiv |\Psi\rangle \otimes |\Phi\rangle$$

state for left   state for right

left qubit $\otimes$ right qubit

$|\Psi\rangle \quad |\Phi\rangle$

Let U be an operator (matrix) for left
V " " " " right

Then $U|\Psi\rangle$ is a state for left
$V|\Phi\rangle$ " " " right

Thus

$$(U|\Psi\rangle) \otimes (V|\Phi\rangle)$$

is again a tensor product state. We express this as:

$$\boxed{\underbrace{(U \otimes V)}_{\text{joint left/right operator}} \underbrace{(|\Psi\rangle \otimes |\Phi\rangle)}_{\text{joint left/right state}} := U|\Psi\rangle \otimes V|\Phi\rangle}$$

which serves as a definition of the tensor product of two operators. We can show that this satisfies:

1) $U \otimes (V_1 + V_2) = U \otimes V_1 + U \otimes V_2$
2) $(U_1 + U_2) \otimes V = U_1 \otimes V + U_2 \otimes V$

To make this more concrete we can show:

If U and V are each represented in the $\{|0\rangle, |1\rangle\}$ basis as:

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \qquad V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$$

then:

$$U \otimes V \sim \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_2 v_1 & u_2 v_2 \\ u_1 v_3 & u_1 v_4 & u_2 v_3 & u_2 v_4 \\ u_3 v_1 & u_3 v_2 & u_4 v_1 & u_4 v_2 \\ u_3 v_3 & u_3 v_4 & u_4 v_3 & u_4 v_4 \end{pmatrix}$$

Thus the tensor product takes and $n \times n$ matrix and a $m \times m$ matrix and produces a $mn \times mn$ matrix. To prove the above consider the upper left entry. This is

$$(1\ 0\ 0\ 0)\,(U \otimes V) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

But $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |0\rangle \otimes |0\rangle$

gives
$$\begin{aligned} \langle 0|\langle 0| (U \otimes V)\, |0\rangle |0\rangle &= \langle 0|\langle 0| (U|0\rangle \otimes V|0\rangle) \\ &= \langle 0|U|0\rangle \langle 0|V|0\rangle \\ &= u_1 v_1 \end{aligned}$$

All the remaining entries follow in the same way.

Then we can show that for $\{|0\rangle, |1\rangle\}$ measurements on each qubit

$$\hat{P}_{00} = \hat{P}_0 \otimes \hat{P}_0$$

$$\hat{P}_{01} = \hat{P}_0 \otimes \hat{P}_1$$

$$\vdots$$

where $\hat{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\hat{P}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

are measurement operators for individual qubits. Now the operators for measurements on a single qubit are obtained via:

For measurements on the left the operator is $\hat{P}_{00} + \hat{P}_{01}$

$$= \hat{P}_0 \otimes \hat{P}_0 + \hat{P}_0 \otimes \hat{P}_1$$

$$= \hat{P}_0 \otimes (\hat{P}_0 + \hat{P}_1)$$

$$= \hat{P}_0 \otimes \hat{I}$$

So the operators for single qubit measurements are:

Left qubit: $\hat{P}_0 \otimes \hat{I}$, $\hat{P}_1 \otimes \hat{I}$

Right " $\hat{I} \otimes \hat{P}_0$, $\hat{I} \otimes \hat{P}_1$

2 Projectors for two qubits

Consider two qubits and suppose that the left is measured in the basis $\{|0\rangle, |1\rangle\}$ while the right is measured in the basis

$$\left\{ \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right), \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle\right) \right\}$$

a) Determine the four projectors for the all pairs of outcomes.

b) Determine the projectors for measurements on the left qubit regardless of those on the right.

c) Determine the projectors for measurements on the right qubit regardless of those on the left.

d) Suppose that the pair is initially in the state

$$|\Psi\rangle = \frac{1}{2} \left(|00\rangle - |01\rangle + |10\rangle + |11\rangle\right).$$

and the left qubit is measured in the basis $\{|0\rangle, |1\rangle\}$. Determine the probability with which either outcome occurs and the two states after measurement.

Answer:

For left

outcome	projector
0	$\hat{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
1	$\hat{P}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

For right

outcome	projector
+	$P_+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
−	$P_- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

a) $\hat{P}_{0+} = \hat{P}_0 \otimes \hat{P}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{2}$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{P}_{0-} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{P}_{1+} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \end{pmatrix}$$

$$\hat{P}_{1-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix} \end{pmatrix}$$

b) $$\hat{P}_0 \otimes \hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{P}_1 \otimes \hat{I} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

c) $$\hat{I} \otimes \hat{P}_+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\hat{I} \otimes \hat{P}_- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

d) $$\text{Prob}(0) = \langle \Psi | \hat{P}_0 \otimes \hat{I} | \Psi \rangle = \frac{1}{2}(1 \; -1 \; 1 \; 1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}\frac{1}{2}$$

$$= \frac{1}{4}(1 \; -1 \; 1 \; 1)\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}$$

state after: $$\frac{\hat{P}_0 \otimes \hat{I} |\Psi\rangle}{\sqrt{\langle \Psi | \hat{P}_0 \otimes \hat{I} | \Psi \rangle}} = \frac{1}{1/\sqrt{2}} \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle)$$

$$= 1/\sqrt{2}\, |0\rangle \otimes (|0\rangle - |1\rangle)$$

For outcome 1 $\quad \hat{P}_1 \otimes \hat{I} \, |\Psi\rangle$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{2} = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

So $\quad \text{Prob}(1) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \frac{1}{2} = \frac{1}{2}$

State after $= \dfrac{\hat{P}_1 \otimes \hat{I} \, |\Psi\rangle}{\sqrt{\langle \Psi | \hat{P}_1 \otimes \hat{I} \, |\Psi\rangle}} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

$$= \frac{1}{\sqrt{2}} \left(|10\rangle + |11\rangle \right)$$

$$= \frac{1}{\sqrt{2}} |1\rangle \otimes \left(|0\rangle + |1\rangle \right)$$