Phys 396 2018

Lecture 7

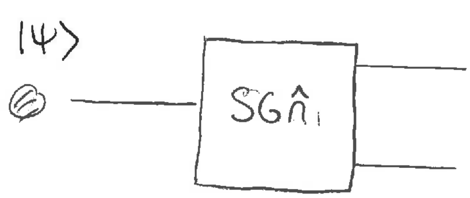
Tues: Turn in HW 3

Thurs: Seminar

Next week Friday - picnic?

Quantum states and measurements

We have seen that for spin-½ systems one meaning of states is that they give measurement statistics. A typical situation is:



outcome	state	prob
$S_n = +\hbar/2$	$\lvert+\hat{n}\rangle$	$\lvert\langle +\hat{n}\lvert\psi\rangle\rvert^2$
$S_n = -\hbar/2$	$\lvert-\hat{n}\rangle$	$\lvert\langle -\hat{n}\lvert\psi\rangle\rvert^2$

Then in order to compute such inner products, we need a representation for states $|+\hat{n}\rangle, |-\hat{n}\rangle$. This is:

For any unit vector $\hat{n}$ that is described using spherical co-ordinates θ, ϕ these states are:

$$|+\hat{n}\rangle = \cos(\theta/2)\,|+\hat{z}\rangle + e^{i\phi}\sin(\theta/2)\,|-\hat{z}\rangle$$

$$|-\hat{n}\rangle = \sin(\theta/2)\,|+\hat{z}\rangle - e^{i\phi}\sin(\theta/2)\,|-\hat{z}\rangle$$

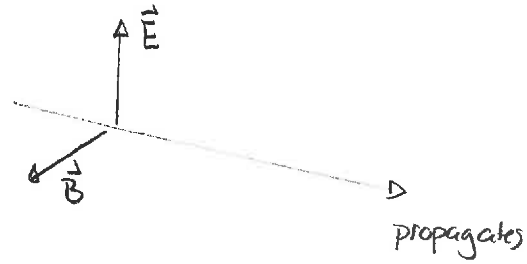
We now aim to extend this to other physical systems. The example that we will consider is single photons and specifically their polarization states. We will:

- review classical polarization
- adapt this to quantum systems.

Polarization in classical electromagnetism.

Classical electromagnetic theory predicts that light is an electromagnetic wave consisting of electric + magnetic fields that vary in space + time. By starting with Maxwell's equations in a vacuum one can show that:

1) waves of electric + magnetic fields exist.
2) the electric field and magnetic fields are perpendicular to each other and perpendicular to the direction of propagation
3) the field magnitudes are related by
$$|\vec{B}| = \frac{1}{c}|\vec{E}|$$



Thus, in order to describe an electromagnetic wave, we only need

1) a direction of propagation
2) the electric field

A particular important class of such waves are sinusoidal plane waves. Examples of such waves that propagate along the z-axis are:

Example a) $\vec{E} = \vec{E}_0 \cos(kz-\omega t)$

b) $\vec{E} = \vec{E}_0 \sin(kz-\omega t)$

A more convenient representation of these is via complex exponentials, in which case a generic form for the wave is:

A generic plane electromagnetic wave that propagates along the z axis is:

$$\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$$

where $k = 2\pi/\lambda$ is the wavenumber

$\omega = 2\pi f$ " " frequency

$\vec{E}_0$ is independent of z and t.

This is set up so that the observed electric field is attained from the complex exponential via:

$\vec{E}$ is complex representation $\longrightarrow$ Observed field is $\mathrm{Re}(\vec{E})$

Linear polarization

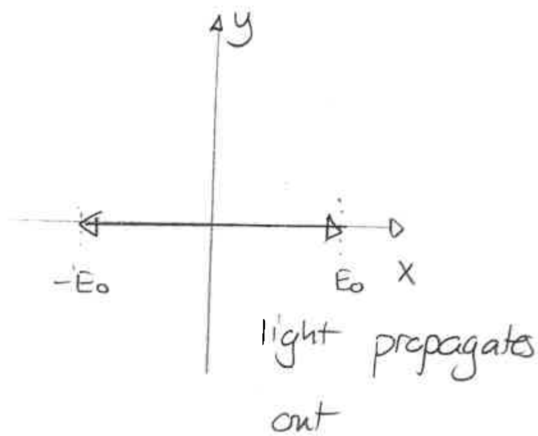
Consider the following possibility for the electric field:

$$\vec{E} = E_0 \hat{x} e^{i(kz-\omega t)}$$

Then the real part of this is

$$E_0 \hat{x} \cos(kz-\omega t)$$

and the electric field oscillates along the $\hat{x}$ axis. This is called horizontally polarized light, and is one example of linearly polarized light.

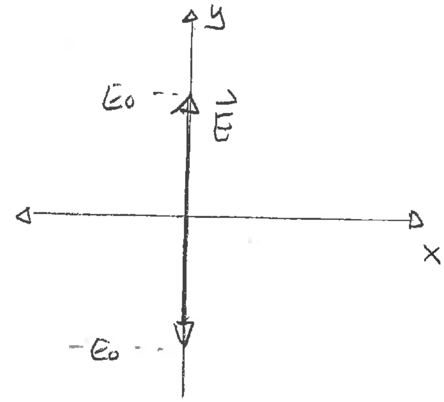


Another example would be if $\vec{E}_0 = E_0 \hat{y}$. Then

$$\vec{E} = E_0 \hat{y} e^{i(kx-\omega t)}$$

gives a real field of $E_0 \hat{y} \cos(kz-\omega t)$ and this oscillates purely along the $\hat{y}$ axis. This is called vertically polarized.

Most light sensors will not distinguish between these. We will see that certain optical elements can do this



Exercise: Consider the following complex amplitudes. Describe the result real electric field at a given location as time passes.

a) $\vec{E}_0 = E_{0x}\hat{x} + E_{0y}\hat{y}$

b) $\vec{E}_0 = \frac{E_0}{\sqrt{2}}\left(\hat{x} + e^{i\pi/2}\hat{y}\right)$

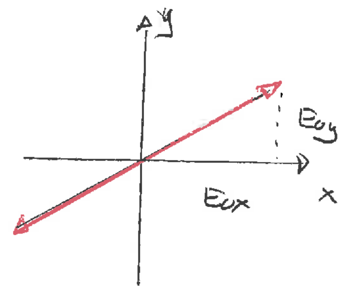
c) $\vec{E}_0 = \left(E_{0x}\hat{x} + e^{i\pi/2}E_{0y}\hat{y}\right)$

where E_{0x}, E_{0y}, are real...

Answer: In all cases we need the real part of $\vec{E}_0 e^{i(kz-\omega t)}$

a) $$Re\left(\vec{E}_0 e^{i(kz-\omega t)}\right) = \left(E_{0x}\hat{x} + E_{0y}\hat{y}\right)\cos(kz-\omega t)$$

This is linearly polarized along the indicated axis

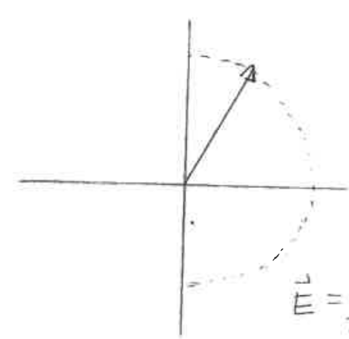


b) $$\vec{E} = \frac{E_0}{\sqrt{2}}\hat{x}\, e^{i(kz-\omega t)} + \frac{E_0}{\sqrt{2}}\hat{y}\, e^{i(kz-\omega t+\pi/2)}$$

$$Re(\vec{E}) = \frac{E_0}{\sqrt{2}}\cos(kz-\omega t)\hat{x} + \frac{E_0}{\sqrt{2}}\cos(kz-\omega t+\pi/2)\hat{y}$$

$$= \frac{E_0}{\sqrt{2}}\left(\cos(kz-\omega t)\hat{x} - \sin(kz-\omega t)\hat{y}\right)$$

This gives a field that rotates about the z-axis in a counterclockwise sense. This is left circular polarization.



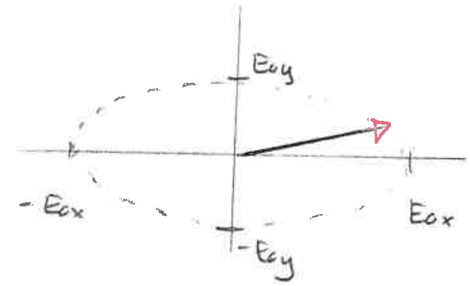
at $z=0$

$$\vec{E} = \frac{E_0}{\sqrt{2}}\cos(\omega t)\hat{x} + \frac{E_0}{\sqrt{2}}\sin(\omega t)\hat{y}$$

c) Similar to b)

$$\vec{E} = E_{0x} \cos(kz-\omega t) - E_{0y} \sin(kz-\omega t)$$

traces an ellipse that rotates counterclockwise about z. This is an example of elliptical polarization.

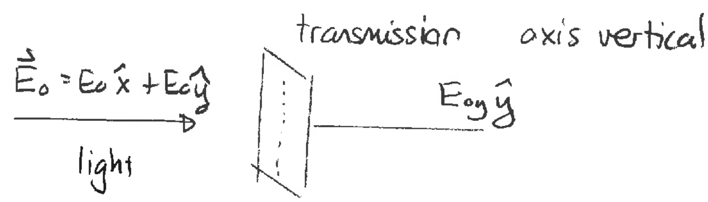


Detecting polarization states

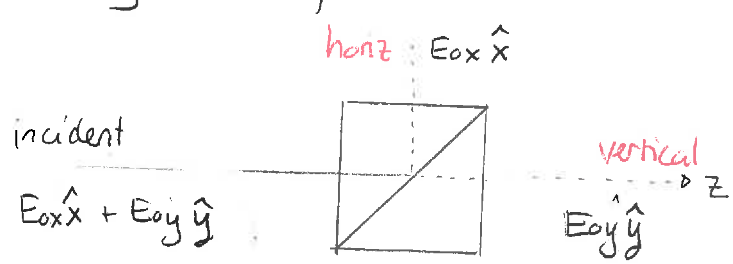
How could we detect the polarization state of light? There are various possibilities.

1) linear polarizing filters absorb the component of the field perpendicular to a transmission axis and transmit the component parallel to the transmission axis.

This could be regarded as a type of measurement although it destroys the system of light if it is perpendicularly polarized



2) polarizing beam splitter. A beam splitter transmits some light and reflects the rest. A polarizing beam splitter is sensitive to the polarization. This can be used to detect and transmit as it does not absorb light



Intensity + polarization

In classical optics intensity is the rate at which the wave transmits energy per second per unit area perpendicular to its propagation direction. Various results in electromagnetism yield:

> If $\vec{E}$ is the complex representation of a wave then the intensity of this light is:
>
> $$I = c\epsilon_0 \vec{E}^* \cdot \vec{E}/2$$

Exercise: Suppose that the following electric fields are incident on a PBS. Determine the fraction of intensity reflected + transmitted in each case:

a) Linearly polarized $\quad \vec{E} = (E_{ox}\hat{x} + E_{oy}\hat{y})\, e^{i(kz-\omega t)}$

b) Circularly " $\quad \vec{E} = \frac{E_o}{\sqrt{2}}(\hat{x} + e^{\pm i\pi/2}\hat{y})\, e^{i(kz-\omega t)}$

Answer: a) Incident intensity $\quad I_i = \frac{c\epsilon_o}{2}(E_{ox}^2 + E_{oy}^2)$

Reflected (horiz) field $\quad \vec{E} = E_{ox}\hat{x}\, e^{i(kz-\omega t)}$

$$I_r = \frac{c\epsilon_o}{2}(E_{ox})^2 \Rightarrow \text{fraction} = \frac{E_{ox}^2}{E_{ox}^2 + E_{oy}^2}$$

Transmitted (vert) $\quad I_t = \frac{c\epsilon_o}{2}(E_{oy})^2 \Rightarrow \text{"} \quad = \frac{E_{oy}^2}{E_{ox}^2 + E_{oy}^2}$

b) Incident intensity $\quad I_i = \frac{c\epsilon_o}{2} E_o^2$

Reflected field (horiz) $\quad \frac{E_o \hat{x}}{\sqrt{2}} e^{i(kz-\omega t)}$

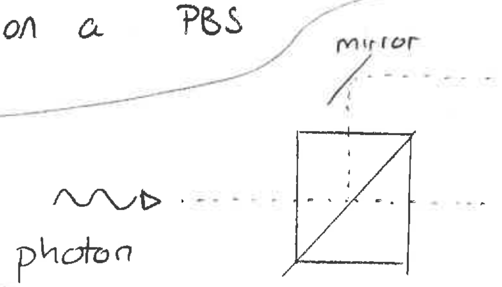
$$I_r = \frac{c\epsilon_o}{2} E_o^2 \Rightarrow \text{fraction} = 1/2$$

Transmitted field $\quad \frac{E_o \hat{y}}{\sqrt{2}} e^{i(kz-\omega t)}$

$$I_t = \frac{c\epsilon_o}{2} E_o^2 \Rightarrow \text{fraction} = 1/2$$

Single photon polarization

Single photon also possess polarization states even though they cannot be described in terms of a wave. Consider such photons incident on a PBS



outcome	state
horiz	$\lvert\rightarrow\rangle$
vertical	$\lvert\uparrow\rangle$

There are two special states relative to this:

$\lvert\rightarrow\rangle \Leftrightarrow$ reflected by PBS with certainty (horizontal)

$\lvert\uparrow\rangle \Leftrightarrow$ transmitted " " " " (vertical

When subjected to another PBS whatever is reflected will again be reflected with certainty so the state after is $\lvert\rightarrow\rangle$. Thus these have the same properties as $\lvert+\hat{z}\rangle$ and $\lvert-\hat{z}\rangle$ states do for spin-½ SG$\hat{z}$ measurements.

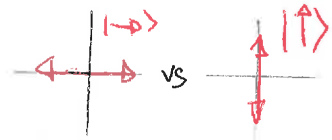
General rules for measurement imply that these kets form a basis. Thus any single photon polarization state must be able to be described as

$$\lvert\psi\rangle = a_x\lvert\rightarrow\rangle + a_y\lvert\uparrow\rangle$$

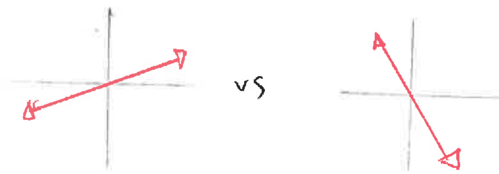
where a_x, a_y are complex. Normalization requires that $\lvert a_x\rvert^2 + \lvert a_y\rvert^2 = 1$.

We need general rules for deciding measurement outcomes. These include the following measurements:

a) horiz / vertical (via PBS)

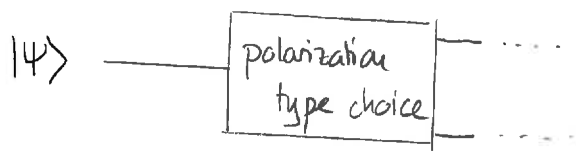


b) arbitrary orthogonal (via "modified" PBS) linear polarizations



c) L vs R circular

We would like the same mathematics



outcome	state	prob
	$\vert\varphi_1\rangle$	$\vert\langle\varphi_1\vert\psi\rangle\vert^2$
	$\vert\varphi_2\rangle$	$\vert\langle\varphi_2\vert\psi\rangle\vert^2$

and we just need the relevant mathematics. In this case the rule is

If $\vec{E}_0 = E_{0x}\,\hat{x} + E_{0y}\,e^{i\phi}\,\hat{y}$ describes the classical polarization state then the associated single photon quantum state is:

$$|\psi\rangle = \frac{1}{\sqrt{E_{0x}^2 + E_{0y}^2}}\left\{E_{0x}\,|\rightarrow\rangle + E_{0y}\,e^{i\phi}\,|\uparrow\rangle\right\}$$

Exercise

a) Determine expressions for the state of a photon which is linearly polarized along the axis $45°$ between $+\hat{x}$ and $+\hat{y}$. Call this $|\nearrow\rangle$

b) Repeat for the state, of a photon which is linearly polarized along the axis $45°$ between $+\hat{x}$ and $-\hat{y}$. Denote $|\searrow\rangle$

c) Repeat for the L and R circularly polarized states.

d) A photon in state $|\nearrow\rangle$ is subjected to a horiz/vert measurement. Determine probabilities of outcomes

e) - - - - - - $|\searrow\rangle$ - - - - -

f) A photon in the $|L\rangle$ state is subjected to a arbitrary linear polarization measurement. Determine probabilities of outcomes

Answer:

a) $\phi = 0 \qquad E_{0x} = E_{0y} \quad \Rightarrow \quad |\nearrow\rangle = \frac{1}{\sqrt{2}}\{|\rightarrow\rangle + |\uparrow\rangle\}$

b) $\phi = 0 \qquad E_{0x} = -E_{0y} \quad \Rightarrow \quad |\searrow\rangle = \frac{1}{\sqrt{2}}\{|\rightarrow\rangle - |\uparrow\rangle\}$

c) $\phi = \pi/2 \qquad E_{0x} = E_{0y} \quad \Rightarrow \quad |L\rangle = \frac{1}{\sqrt{2}}\{|\rightarrow\rangle + i|\uparrow\rangle\}$

$\phi = -\pi/2 \qquad E_{0x} = E_{0y} \quad \Rightarrow \quad |R\rangle = \frac{1}{\sqrt{2}}\{|\rightarrow\rangle - i|\downarrow\rangle\}$

d) $\text{Prob (horiz)} = |\langle\leftrightarrow|\nearrow\rangle|^2$

$$\langle\rightarrow|\nearrow\rangle = \langle\rightarrow|\left(\frac{1}{\sqrt{2}}|\rightarrow\rangle + |\uparrow\rangle\right) = \frac{1}{\sqrt{2}}\underbrace{\langle\rightarrow|\rightarrow\rangle}_{=1} + \frac{1}{\sqrt{2}}\underbrace{\langle\rightarrow|\uparrow\rangle}_{=0}$$

$$= \frac{1}{\sqrt{2}}$$

$\Rightarrow \text{Prob (horiz)} = 1/2$

Likewise $\text{Prob (vert)} = 1/2$

e) Similar to d) $\quad \text{Prob (horiz)} = |\langle \leftrightarrow | \nearrow \rangle|^2$

and $\quad \langle \leftrightarrow | \nearrow \rangle = \frac{1}{\sqrt{2}} \Rightarrow \text{Prob} = 1/2$

$$\text{Prob(vert)} = |\langle \updownarrow | \nearrow \rangle|^2 = 1/2$$

f) Let $\quad |\varphi_1\rangle = \frac{1}{\sqrt{E_{ox}^2 + E_{oy}^2}} \left\{ E_{ox} |\rightarrow\rangle + E_{oy} |\uparrow\rangle \right\}$

$$|\varphi_2\rangle = \frac{1}{\sqrt{\quad}} \left\{ E_{oy} |\rightarrow\rangle - E_{ox} |\downarrow\rangle \right\}$$

represent the two polarizations

$$\text{Prob (outcome 1)} = |\langle \varphi_1 | L \rangle|^2$$

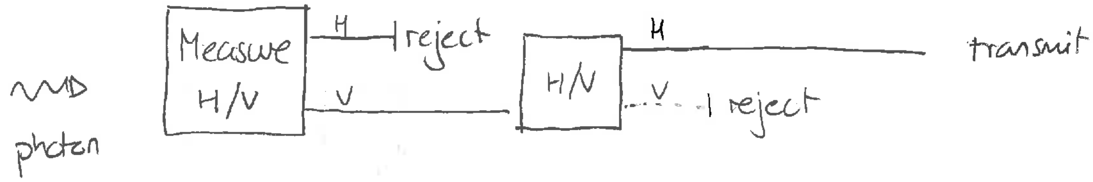
$$\langle \varphi_1 | L \rangle = \frac{1}{\sqrt{E_{ox}^2 + E_{oy}^2}} \left(\frac{1}{\sqrt{2}} E_{ox} + \frac{i}{\sqrt{2}} E_{oy} \right)$$

$$\text{Prob (outcome 1)} = \frac{1}{E_{ox}^2 + E_{oy}^2} \left(\frac{1}{2} E_{ox}^2 + \frac{1}{2} E_{oy}^2 \right) = \frac{1}{2}$$

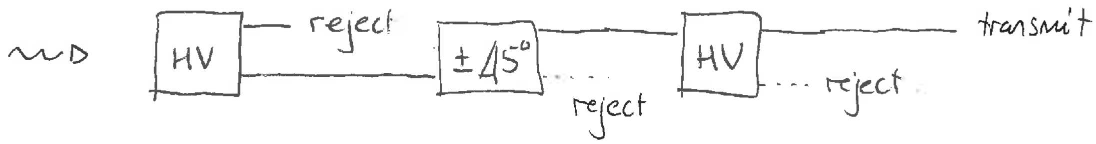
Likewise Prob (outcome 2) $= 1/2$

Exercise: Consider the two experiments:

a)



b)



Determine probability that photon passes each, given initially in state $|\psi\rangle$

Answer: a) After 1^{st} state is $|\uparrow\rangle$ but this is blocked by second

b) After 1st state is $|\uparrow\rangle$. This passes upper arm of second with prob $= 1/2$ and in state $|\nearrow\rangle$. This passes upper arm of third with prob $= 1/2$. Total probability is.

$$\frac{1}{2} \cdot \frac{1}{2} |\langle\uparrow|\psi\rangle|^2 = \frac{1}{4} |\langle\uparrow|\psi\rangle|^2$$