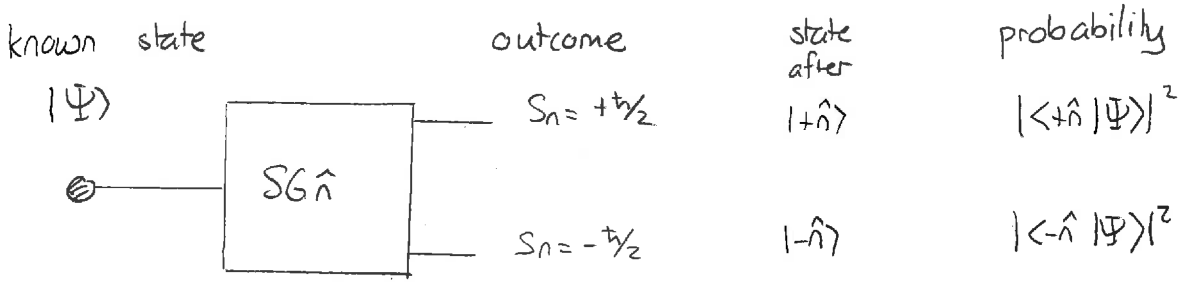
Fri: SPS

Tues: HW by 5pm → Recent last two years

Read R $\begin{cases} 2.1.1, 2.1.2 & 2.5.2 \text{ Bloch sphere.} \\ 3.1.2 \end{cases}$

States and measurements for spin-½ particles

We have now seen the following general scheme for determining measurement outcomes given a known input state for spin-½ particles:



We can do the calculations by expressing any state in terms of a given fixed basis, usually $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$. Specifically

If $\hat{n}$ is a real three dimensional unit vector with spherical co-ordinates θ, ϕ then

$$|+\hat{n}\rangle = \cos\left(\frac{\theta}{2}\right)|+\hat{z}\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-\hat{z}\rangle$$

$$|-\hat{n}\rangle = \sin(\theta/2)|+\hat{z}\rangle - e^{i\phi}\cos(\theta/2)|-\hat{z}\rangle$$

Bra vectors

We now describe a mathematical technique that streamlines many calculations, particularly those of an inner product type. Consider two ket vectors

$$|\Psi\rangle = a_+|+\hat{z}\rangle + a_-|-\hat{z}\rangle$$

$$|\Phi\rangle = b_+|+\hat{z}\rangle + b_-|-\hat{z}\rangle$$

Then:

$$\langle\Phi|\Psi\rangle = b_+^* a_+ + b_-^* a_-$$

can be expressed in terms of column and row vectors as:

$$\langle\Phi|\Psi\rangle = \begin{pmatrix} b_+^* & b_-^* \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

If we represent $|\Psi\rangle$ by the column vector $\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$ then we can create a new type of mathematical object:

$$\langle\Phi| \leadsto \begin{pmatrix} b_+^* & b_-^* \end{pmatrix}$$

This is a row vector. The set of all row vectors constitutes a vector space if we define addition:

$$(u_1 \quad u_2) + (v_1 \quad v_2) := (u_1+v_1 \quad u_2+v_2)$$

and multiplication

$$\alpha(u_1 \quad u_2) = (\alpha u_1 \quad \alpha u_2)$$

Observe that row vectors are distinct from column vectors in the sense that, for example, one cannot add a row vector to a column vector.

In quantum physics objects such as $\langle\Phi|$ are called bra vectors. Mathematically a bra vector is a map that takes a ket vector as an input and produces a complex scalar

$$\text{ket } |\Psi\rangle \longrightarrow \boxed{\begin{array}{c}\text{"bra" machine}\\ \langle\Phi|\end{array}} \longrightarrow \text{complex number } \langle\Phi|\Psi\rangle.$$

This operation is linear so

$$\alpha_1|\Psi_1\rangle + \alpha_2|\Psi_2\rangle \longrightarrow \boxed{\langle\Phi|} \longrightarrow \alpha_1\langle\Phi|\Psi_1\rangle + \alpha_2\langle\Phi|\Psi_2\rangle$$

One can provide a vector space structure to bra vectors by adding + multiplying linear operations.

The inner product on the space of kets allows for a unique association between kets and bras. Specifically:

> Given any ket $|\Phi\rangle$, the associated bra $\langle\Phi|$ is a linear map on the space of kets such that for any ket $|\Psi\rangle$
>
> $$\langle\Phi| \begin{pmatrix}\text{acting on}\\ |\Psi\rangle\end{pmatrix} = \langle\Phi|\Psi\rangle$$

Note that we can define the sum of two bras in terms of its action on a ket:

Given $\langle \Phi_1 |$ and $\langle \Phi_2 |$ and β_1, β_2 then $\beta_1 \langle \Phi_1 | + \beta_2 \langle \Phi_2 |$ is a bra whose action on any ket $|\Psi\rangle$ is:

$$\left(\beta_1 \langle \Phi_1 | + \beta_2 \langle \Phi_2 | \right) |\Psi\rangle = \beta_1 \langle \Phi_1 | \Psi \rangle + \beta_2 \langle \Phi_2 | \Psi \rangle$$

One particularly useful case is:

For any $\hat{n}$, the bras $\langle \pm\hat{n}|$ satisfies

$$\langle +\hat{n}|+\hat{n}\rangle = 1$$

$$\langle +\hat{n}|-\hat{n}\rangle = 0$$

$$\langle -\hat{n}|-\hat{n}\rangle = 1$$

$$\langle -\hat{n}|+\hat{n}\rangle = 0$$

Another useful result is:

If $|\Phi\rangle = \alpha_1|\varphi_1\rangle + \alpha_2|\varphi_2\rangle$ then $\langle\Phi| = \alpha_1^*\langle\varphi_1| + \alpha_2^*\langle\varphi_2|$

<u>Proof:</u> Consider any ket $|\Psi\rangle$ and $\langle\Phi|\Psi\rangle$

Then by inner product rules, $\langle\Phi|\Psi\rangle = (\langle\Psi|\Phi\rangle)^*$

by linearity $\langle\Psi|\Phi\rangle = \alpha_1\langle\Psi|\varphi_1\rangle + \alpha_2\langle\Psi|\varphi_2\rangle$

$$\Rightarrow \quad \langle\Phi|\Psi\rangle = \left(\alpha_1\langle\Psi|\varphi_1\rangle + \alpha_2\langle\Psi|\varphi_2\rangle\right)^*$$

$$= \alpha_1^*\left(\langle\Psi|\varphi_1\rangle\right)^* + \alpha_2^*\left(\langle\Psi|\varphi_2\rangle\right)^*$$

$$= \alpha_1^*\langle\varphi_1|\Psi\rangle + \alpha_2^*\langle\varphi_2|\Psi\rangle$$

$$= \left\{\alpha_1^*\langle\varphi_1| + \alpha_2^*\langle\varphi_2|\right\}|\Psi\rangle.$$

This is true for all $|\Psi\rangle$. ∎

Finally, included in the above is:

1 Bra vectors

Consider the kets

$$
\begin{aligned}
|\Psi\rangle &= \frac{3}{5}\,|+\hat{z}\rangle + \frac{4i}{5}\,|-\hat{z}\rangle \\
|\varphi_1\rangle &= \frac{5i}{13}\,|+\hat{z}\rangle + \frac{12}{13}\,|-\hat{z}\rangle \\
|\varphi_2\rangle &= \frac{1+2i}{\sqrt{10}}\,|+\hat{z}\rangle + \frac{1-2i}{\sqrt{10}}\,|-\hat{z}\rangle
\end{aligned}
$$

a) Determine expressions for $|\varphi_1\rangle$ and $|\varphi_2\rangle$ in terms of $\langle +\hat{\mathbf{z}}|$ and $\langle -\hat{\mathbf{z}}|$.

b) Use these expressions to compute $\langle \varphi_i|\Psi\rangle$ for $i = 1, 2$.

Answer:

a)
$$\langle \varphi_1| = \left(\frac{5i}{13}\right)^* \langle +\hat{z}| + \left(\frac{12}{13}\right)^* \langle -\hat{z}|$$

$$= \frac{-5i}{13}\langle +\hat{z}| + \frac{12}{13}\langle -\hat{z}|$$

$$\langle \varphi_2| = \left(\frac{1+2i}{\sqrt{10}}\right)^* \langle +\hat{z}| + \left(\frac{1-2i}{\sqrt{10}}\right)^* \langle -\hat{z}|$$

$$= \frac{1-2i}{\sqrt{10}}\langle +\hat{z}| + \frac{1+2i}{\sqrt{10}}\langle -\hat{z}|$$

b)
$$\langle \varphi_1|\Psi\rangle = \left(-\frac{5i}{13}\langle +\hat{z}| + \frac{12}{13}\langle -\hat{z}|\right)\left(\frac{3}{5}|+\hat{z}\rangle + \frac{4i}{5}|-\hat{z}\rangle\right)$$

$$= -\frac{15i}{65}\underbrace{\langle +\hat{z}|+\hat{z}\rangle}_{1} - \frac{5i}{65}4i\,\cancel{\langle +\hat{z}|-\hat{z}\rangle} + \frac{36}{65}\cancel{\langle -\hat{z}|+\hat{z}\rangle} + \frac{48i}{65}\underbrace{\langle -\hat{z}|-\hat{z}\rangle}_{1}$$

$$= \frac{33i}{65}$$

$$\langle \varphi_2|\Psi\rangle = \left[\left(\frac{1-2i}{\sqrt{10}}\right)\langle +\hat{z}| + \frac{1+2i}{\sqrt{10}}\langle -\hat{z}|\right]\left[\frac{3}{5}|+\hat{z}\rangle + \frac{4i}{5}|-\hat{z}\rangle\right]$$

$$= \frac{3-6i}{5\sqrt{10}} + \frac{4i-8}{5\sqrt{10}} = \frac{-5-2i}{5\sqrt{10}}$$

We can also represent bra vectors in terms of row vectors and basic row vectors. Recall

$$|+\hat{z}\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\hat{z}\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is a standard representation of kets via column vectors. Now we can represent bra vectors as rows. So

$$\langle +\hat{z}| \leadsto (b_+ \;\; b_-)$$

But $\langle +\hat{z}|+\hat{z}\rangle = 1 \Rightarrow \quad (b_+ \;\; b_-)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = b_+ \Rightarrow b_+ = 1$

$\langle +\hat{z}|-\hat{z}\rangle = 0 \Rightarrow \quad (b_+ \;\; b_-)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = b_- \Rightarrow b_- = 0$

Thus

$$\boxed{\begin{aligned} \langle +\hat{z}| &\leftrightarrow (1 \;\; 0) \\ \langle -\hat{z}| &\leftrightarrow (0 \;\; 1) \end{aligned}}$$

and any bra is represented via:

$$b_+ \langle +\hat{z}| + b_- \langle -\hat{z}| \leftrightarrow (b_+ \;\; b_-)$$

We can then use row and column vector operations to determine the action of bra vectors on kets.

Transpose and adjoint operations.

The process of associating a bra with a ket is comparable to that of associating a row vector with a column vector. A convenient way to represent this is via matrix transposition and complex conjugation. First consider the transpose operation:

Given any $m \times n$ matrix A, the transpose A^T is a $n \times m$ matrix with entries $(A^T)_{ij} = A_{ji}$

For example if

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \end{pmatrix}$$

then

$$A^T = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \\ A_{13} & A_{23} \\ A_{14} & A_{24} \end{pmatrix}$$

One can prove that:

For any matrices A, B

$$(A+B)^T = A^T + B^T$$

$$(\lambda A)^T = \lambda A^T$$

$$(AB)^T = B^T A^T$$

$$(A^T)^T = A$$

We can see that this will map a column vector to a row vector

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T = (u_1 \quad u_2)$$

This appears to be related to the ket $\rightarrow$ bra correspondence. However, it lacks the complex conjugate. Thus we can define the complex conjugate transpose or adjoint via:

> Given an $m \times n$ matrix A with entries A_{ij}, the <u>adjoint</u> $A^\dagger$ is an $n \times m$ matrix with entries $(A^\dagger)_{ij} = A^*_{ji}$

Thus if

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

then

$$A^\dagger = \begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix}$$

This satisfies:

$$\boxed{\begin{aligned} (A+B)^\dagger &= A^\dagger + B^\dagger \\ (\lambda A)^\dagger &= \lambda^* A^\dagger \\ (AB)^\dagger &= B^\dagger A^\dagger \\ (A^\dagger)^\dagger &= A \end{aligned}}$$

2 Matrix adjoints

Consider the two matrices

$$A = \begin{pmatrix} 2 & 2 \\ -i & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1+i & 2 \\ 1-i & 1 & 0 \end{pmatrix}$$

a) Determine $A^\dagger$.

b) Determine $B^\dagger$.

c) Determine AB.

d) Determine $(AB)^\dagger$ and show that this equals $B^\dagger A^\dagger$.

Answer

a) $A^\dagger = \begin{pmatrix} 2 & i \\ 2 & 1 \end{pmatrix}$

b) $B^\dagger = \begin{pmatrix} 0 & 1+i \\ 1-i & 1 \\ 2 & 0 \end{pmatrix}$

c) $AB = \begin{pmatrix} 2 & 2 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 0 & 1+i & 2 \\ 1-i & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2-2i & 4+2i & 4 \\ 1-i & 2-i & -2i \end{pmatrix}$

d) $(AB)^\dagger = \begin{pmatrix} 2+2i & 1+i \\ 4-2i & 2+i \\ 4 & 2i \end{pmatrix}$

$B^\dagger A^\dagger = \begin{pmatrix} 0 & 1+i \\ 1-i & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & i \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2+2i & 1+i \\ 4-2i & 2+i \\ 4 & 2i \end{pmatrix}$

$= (AB)^\dagger$

In terms of row + column vectors we can express kets and associated bras using:

If $|\Psi\rangle = c_+ |+\hat{z}\rangle + c_- |-z\rangle$ then

$$|\Psi\rangle \rightsquigarrow \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

But $\langle\Psi| = c_+^* \langle+\hat{z}| + c_-^* \langle-\hat{z}| = (c_+^* \;\; c_-^*) = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}^\dagger$

Thus we say

$$\boxed{\langle\Psi| = |\Psi\rangle^\dagger}$$

Here all the usual rules of the adjoint apply.

So if

$$|\Psi\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle$$

then

$$\begin{aligned} \langle\Psi| = \left(|\Psi\rangle\right)^\dagger &= \left(a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle\right)^\dagger \\ &= a_+^* |+\hat{z}\rangle^\dagger + a_-^* |-\hat{z}\rangle^\dagger \\ &= a_+^* \langle+\hat{z}| + a_-^* \langle-\hat{z}| \end{aligned}$$

Special cases are $|+\hat{z}\rangle^\dagger = \langle+\hat{z}|$
$|-\hat{z}\rangle^\dagger = \langle-\hat{z}|$

3 Bras and kets in probability calculations

A spin-1/2 particle is initially in the state

$$|\Psi\rangle = \frac{1+i}{2}\,|+\hat{z}\rangle + \frac{1-i}{2}\,|-\hat{z}\rangle\,.$$

The particle is subjected to an SG $\hat{y}$ measurement.

a) Determine expressions for the kets associated with the SG $\hat{y}$ measurement in terms of $\{|+\hat{z}\rangle\,, |-\hat{z}\rangle\}$.

b) Use the adjoint operation to construct bra vectors from these kets and to determine the probabilities of the two measurement outcomes.

Answer: a) We need $|\pm\hat{y}\rangle$. For $\hat{y}$ $\theta = \pi/2$ $\phi = \pi/2$

$$\Rightarrow \quad |+\hat{y}\rangle = \cos\frac{\pi}{4}|+\hat{z}\rangle + e^{i\pi/2}\sin\pi/4\,|-\hat{z}\rangle$$

$$|+\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$

Similarly

$$|-\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$

b) $$\langle +\hat{y}| = |+\hat{y}\rangle^{\dagger} = \left(\frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle\right)^{\dagger}$$

$$= \frac{1}{\sqrt{2}}|+\hat{z}\rangle^{\dagger} + \left(\frac{i}{\sqrt{2}}\right)^{*}|-\hat{z}\rangle^{\dagger}$$

$$= \frac{1}{\sqrt{2}}\langle +\hat{z}| + \frac{i}{\sqrt{2}}\langle -\hat{z}|$$

$$\Rightarrow \quad \langle +\hat{y}| = \frac{1}{\sqrt{2}}\langle +\hat{z}| - \frac{i}{\sqrt{2}}\langle -\hat{z}|$$

likewise

$$\langle -\hat{y}| = \frac{1}{\sqrt{2}}\langle +\hat{z}| + \frac{i}{\sqrt{2}}\langle -\hat{z}|$$

Now

$$\text{Prob}\left(S_y = +\hbar/2\right) = \left|\langle +\hat{y}|\Psi\rangle\right|^2$$

and

$$\langle +\hat{y}|\Psi\rangle = \left[\frac{1}{\sqrt{2}}\langle +\hat{z}| - \frac{i}{\sqrt{2}}\langle -\hat{z}|\right]\left[\frac{1+i}{2}|+\hat{z}\rangle + \frac{1-i}{2}|-\hat{z}\rangle\right]$$

$$= \frac{1+i}{2\sqrt{2}}\underbrace{\langle +\hat{z}|+\hat{z}\rangle}_{1} + \frac{1-i}{2\sqrt{2}}\underbrace{\cancel{\langle +\hat{z}|-\hat{z}\rangle}}_{0} - \frac{i-1}{2\sqrt{2}}\underbrace{\cancel{\langle -\hat{z}|+\hat{z}\rangle}}_{0} + \frac{-i-1}{2\sqrt{2}}\underbrace{\langle -\hat{z}|-\hat{z}\rangle}_{1}$$

$$= \frac{1}{2\sqrt{2}}\left[1+i-1-i\right] = 0$$

So $\boxed{\text{Prob}\left(S_y = +\hbar/2\right) = 0}$

Then

$$\langle -\hat{y}|\Psi\rangle = \left[\frac{1}{\sqrt{2}}\langle +\hat{z}| + \frac{i}{\sqrt{2}}\langle -\hat{z}|\right]\left[\frac{1+i}{2}|+\hat{z}\rangle + \frac{1-i}{2}|-\hat{z}\rangle\right]$$

$$= \frac{1}{2\sqrt{2}}\left(1+i+i(1-i)\right) = \frac{1}{2\sqrt{2}}(2+2i) = \frac{1+i}{\sqrt{2}}$$

$\Rightarrow$ $\boxed{\text{Prob}\left(S_y = -\hbar/2\right) = 1}$