Thurs: Seminar

Thurs: Read My notes section 3.2 p 38 – 50

Mathematics for spin-½ systems

We have seen that the state of a spin-½ system may be represented by a ket and the set of all kets constitutes a complex vector space. Thus kets can be added to form new kets and multiplied by scalars to form new kets.

The set of all kets is also equipped with an inner product. Thus

Given any two kets $|\Phi\rangle$ and $|\Psi\rangle$ $\xrightarrow[\text{inner product}]{}$ Single complex number $\langle\Phi|\Psi\rangle$

This inner product satisfies

i) If $|\Psi\rangle = \alpha|\varphi_1\rangle + \beta|\varphi_2\rangle$ then

$$\langle\Phi|\Psi\rangle = \alpha\langle\Phi|\varphi_1\rangle + \beta\langle\Phi|\varphi_2\rangle$$

ii) $$\langle\Psi|\Phi\rangle = \left(\langle\Phi|\Psi\rangle\right)^*$$

We then have a physical requirement.

> Given any SG$\hat{n}$ measurement, the two states associated with the two mutually incompatible measurement outcomes (i.e. $|+\hat{n}\rangle$ and $|-\hat{n}\rangle$) are orthogonal and constitute a basis for the set of all states.

We can always arrange for any basis vectors to be normalized. So, for example a possible basis is $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$ and any state can be expressed as:

$$|\Psi\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle.$$

We could represent this in terms of column vectors using

$$|+\hat{z}\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\hat{z}\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and then

$$|\Psi\rangle \leftrightarrow \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

Note that we can use purely algebraic rules + orthonormality of the basis vectors to compute inner products.

Example: Suppose

$$|\Psi\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle$$

$$|\Phi\rangle = b_+ |+\hat{z}\rangle + b_- |-\hat{z}\rangle$$

Then use existing rules to determine $\langle \Phi | \Psi \rangle$

Answer:

$$\langle \Phi | \Psi \rangle = a_+ \langle \Phi | +\hat{z} \rangle + a_- \langle \Phi | -\hat{z} \rangle$$

But $\langle \Phi | +\hat{z} \rangle = \left(\langle +\hat{z} | \Phi \rangle \right)^*$ and

$$\langle +\hat{z} | \Phi \rangle = b_+ \langle +\hat{z} | +\hat{z} \rangle + b_- \langle +\hat{z} | -\hat{z} \rangle$$

But $\langle +\hat{z} | +\hat{z} \rangle = 1$ and $\langle +\hat{z} | -\hat{z} \rangle = 0$. Thus

$$\langle +\hat{z} | \Phi \rangle = b_+$$

Similarly $\langle -\hat{z} | \Phi \rangle = b_-$. So

$$\langle \Phi | +\hat{z} \rangle = a_+ b_+^* + a_- b_-^*$$

Thus we have an important computational rule:

If

$$|\Psi\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle$$

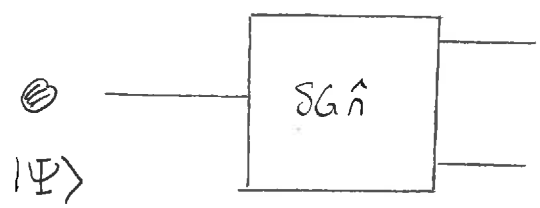
$$|\Phi\rangle = b_+ |+\hat{z}\rangle + b_- |-\hat{z}\rangle$$

then

$$\langle \Phi | \Psi \rangle = b_+^* a_+ + b_-^* a_-$$

Probabilities of measurement outcomes

Suppose that somehow we represent the state of a spin-$\frac{1}{2}$ particle by $|\Psi\rangle$. We want to know how this might be able to predict outcomes of a measurement. So consider a generic type of measurement.



outcome	state after	prob
$S_n = +\hbar/2$	$\lvert +\hat{n}\rangle$	?
$S_n = -\hbar/2$	$\lvert -\hat{n}\rangle$	?

We will usually not be able to predict which measurement outcome occurs with certainty. We therefore aim to predict the probability with which each occurs. The key rule is:

> Given that an SG$\hat{n}$ measurement is performed on a particle in state $|\Psi\rangle$, the probabilities of the two outcomes are:
>
> $$\text{Prob}(S_n = +\hbar/2) = |\langle +\hat{n}|\Psi\rangle|^2$$
>
> $$\text{Prob}(S_n = -\hbar/2) = |\langle -\hat{n}|\Psi\rangle|^2$$
>
> where we use the states $|+\hat{n}\rangle$ and $|-\hat{n}\rangle$ that yield either outcome with certainty

We can immediately see the following. Suppose that

$$|\Psi\rangle = a_+|+\hat{z}\rangle + a_-|-\hat{z}\rangle$$

and one performs a SG $\hat{z}$ measurement. Then we get

$$\text{Prob}\,(S_z = +\hbar/2) = |\langle +\hat{z}|\Psi\rangle|^2$$

But $\quad |+\hat{z}\rangle = 1\,|+\hat{z}\rangle + 0\,|-\hat{z}\rangle \quad$ and so

$$\langle +\hat{z}|\Psi\rangle = a_+$$

Thus $\quad \text{Prob}\,(S_z = +\hbar/2) = |a_+|^2$. Similarly $\text{Prob}\,(S_z = -\hbar/2) = |a_-|^2$

So we get:

If $|\Psi\rangle = a_+|+\hat{z}\rangle + a_-|-\hat{z}\rangle$ then the outcomes of an SG $\hat{z}$ measurement occur with probability

$$\text{Prob}(S_z = +\hbar/2) = |a_+|^2$$

$$\text{Prob}(S_z = -\hbar/2) = |a_-|^2$$

This gives a partial interpretation of the components of $|\Psi\rangle$.

Note that in order for the probabilities to add to 1, we require $|a_+|^2 + |a_-|^2 = 1$. This means that any mathematical ket describing a physical state must be normalized.

$$\boxed{\text{If } |\Psi\rangle \text{ describes a physical state then } \langle\Psi|\Psi\rangle = 1}$$

1 Measurement probabilities

Consider the state

$$|\Psi\rangle = 3i\,|+\hat{z}\rangle + 4\,|-\hat{z}\rangle\,.$$

a) Normalize this state and denote the result by $|\psi\rangle$.

b) Assume that a particle in this state is subjected to a SG $\hat{z}$ measurement. Determine the probabilities of the two outcomes.

c) Consider the state $|\phi\rangle := e^{i\varphi}|\psi\rangle$ where φ is any real number. Express this in terms of $\{|+\hat{z}\rangle\,, |-\hat{z}\rangle\}$. Assume that a particle in this state is subjected to a SG $\hat{z}$ measurement. Determine the probabilities of the two outcomes.

Answer:

a) $\langle\Psi|\Psi\rangle = |3i|^2 + |4|^2 = 25$

So $\frac{1}{5}|\Psi\rangle$ will be normalized. Thus

$$|\psi\rangle = \frac{3i}{5}|+\hat{z}\rangle + \frac{4}{5}|-\hat{z}\rangle$$

b) $\text{Prob}(S_z = \pm\hbar/2) = |\langle\pm\hat{z}|\psi\rangle|^2$

Then $\langle+\hat{z}|\psi\rangle = \frac{3i}{5}$, So

$\text{Prob}(S_z = +\hbar/2) = \left|\frac{3i}{5}\right|^2 \Rightarrow \text{Prob}(S_z = +\hbar/2) = \frac{9}{25}$

Similarly $\langle-\hat{z}|\psi\rangle = \frac{4}{5} \Rightarrow \text{Prob}(S_z = -\hbar/2) = \frac{16}{25}$

c) Here $|\phi\rangle = \frac{3i}{5}e^{i\varphi}|+\hat{z}\rangle + \frac{4}{5}e^{i\varphi}|-\hat{z}\rangle$

Then $\langle+\hat{z}|\phi\rangle = \frac{3i}{5}e^{i\varphi} \Rightarrow \text{Prob}(S_z = +\hbar/2) = \left|\frac{3i}{5}e^{i\varphi}\right|^2$

$\langle-\hat{z}|\phi\rangle = \frac{4}{5}e^{i\varphi}$

$= \left|\frac{3i}{5}\right|^2 \underbrace{|e^{i\varphi}|^2}_{=1}$

$\text{Prob}(S_z = +\hbar/2) = \frac{9}{25}$

Similarly $\text{Prob}(S_z = -\hbar/2) = \frac{16}{25}$

The previous exercise illustrates an example of a general rule.

Suppose that $|\Psi\rangle$ represents the state of a spin-½ particle. Then the probabilities of the outcomes of an SG$\hat{n}$ measurement for this particle are identical to those of a particle in the state $e^{i\varphi}|\Psi\rangle$ where φ is any real number.

Because of this we state that

The mathematical states $|\Psi\rangle$ and $e^{i\varphi}|\Psi\rangle$ represent the same physical state of the particle.

In this context $e^{i\varphi}$ is called a <u>global phase</u>.

So far this has equipped us with the ability to determine the probabilities of the outcomes for an SG$\hat{n}$ measurement provided that we can express the state $|\Psi\rangle$ in terms of $|+\hat{n}\rangle$ and $|-\hat{n}\rangle$. This does not help if we have a particle in some state $|+\hat{m}\rangle$ subjected to an SG$\hat{n}$ measurement. To do this we need to be able to express both states in terms a common basis. The following, consistent with experimental results, shows how to do this

Suppose $\hat{n}$ is any unit vector and in terms of spherical co-ordinates

$$\hat{n} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

Then:

$$|+\hat{n}\rangle = \cos\left(\frac{\theta}{2}\right)|+\hat{z}\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-\hat{z}\rangle$$

$$|-\hat{n}\rangle = \sin\theta/2\,|+\hat{z}\rangle - e^{i\phi}\sin\theta/2\,|-\hat{z}\rangle$$

For a proof, see pg 32-34 of previous notes. So given

$|+\hat{m}\rangle$ — $\boxed{SG\hat{n}}$ Probs = ??

Get spherical angles for $\hat{m}$ and construct $|+\hat{m}\rangle$ in terms of $|+\hat{z}\rangle, |-\hat{z}\rangle$

Get spherical angles for $\hat{n}$ and construct $|+\hat{n}\rangle, |-\hat{n}\rangle$ in terms of $|+\hat{z}\rangle, |-\hat{z}\rangle$

Now calculate $\langle +\hat{m}|+\hat{n}\rangle$ ...

2 Kets in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.

a) Express $|+\hat{x}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.

b) Express $|-\hat{x}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.

c) Express $|+\hat{y}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.

d) Express $|-\hat{y}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.

Answer: a) For $\hat{x}$ $\theta = \pi/2$

$\phi = 0$

So
$$|+\hat{x}\rangle = \cos\left(\frac{\theta}{2}\right)|+\hat{z}\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-\hat{z}\rangle$$
$$|-\hat{x}\rangle = \sin\left(\frac{\theta}{2}\right)|+\hat{z}\rangle - e^{i\phi}\cos\left(\frac{\theta}{2}\right)|-\hat{z}\rangle$$

$\Rightarrow$
$$|+\hat{x}\rangle = \cos(\pi/4)|+\hat{z}\rangle + e^{i0}\sin(\pi/4)|-\hat{z}\rangle$$
$$|-\hat{x}\rangle = \sin(\pi/4)|+\hat{z}\rangle - e^{i0}\cos(\pi/4)|-\hat{z}\rangle$$

$\Rightarrow$
$$|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle$$
$$|-\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{1}{\sqrt{2}}|-\hat{z}\rangle$$

b) Similar but for $\hat{y}$ $\theta = \pi/2$, $\phi = \pi/2$

$\Rightarrow$
$$|+\hat{y}\rangle = \cos\pi/4\,|+\hat{z}\rangle + e^{i\pi/2}\sin\pi/4\,|-\hat{z}\rangle$$
$$|-\hat{y}\rangle = \sin\pi/4\,|+\hat{z}\rangle - e^{+i\pi/2}\cos\pi/4\,|-\hat{z}\rangle$$

$\Rightarrow$
$$|+\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$
$$|-\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$

3 Measurement probabilities

a) A particle in state $|+\hat{x}\rangle$ is subjected to a SG $\hat{z}$ measurement. Use operations on the kets to determine the probabilities of the outcomes.

b) A particle in state $|-\hat{y}\rangle$ is subjected to a SG $\hat{x}$ measurement. Use operations on the kets to determine the probabilities of the outcomes.

Answer: a) Two outcomes $+\hbar/2$, $-\hbar/2$

$$\text{Prob}(S_z = +\hbar/2) = |\langle +\hat{z}|+\hat{x}\rangle|^2$$

$$\langle +\hat{z}|+\hat{x}\rangle = \frac{1}{\sqrt{2}} \Rightarrow \text{Prob}(S_z = +\hbar/2) = \left(\frac{1}{\sqrt{2}}\right)^2$$

$$\Rightarrow \text{Prob}(S_z = +\hbar/2) = \frac{1}{2}$$

$$\text{Prob}(S_z = -\hbar/2) = |\langle -\hat{z}|+\hat{x}\rangle|^2$$

$$\langle -\hat{z}|+\hat{x}\rangle = \frac{1}{\sqrt{2}} \Rightarrow \text{Prob}(S_z = -\hbar/2) = \frac{1}{2}$$

b) $$\text{Prob}(S_x = +\hbar/2) = |\langle +\hat{x}|-\hat{y}\rangle|^2$$

$$|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle$$

$$|-\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$

$$\Rightarrow \langle +\hat{x}|-\hat{y}\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\frac{-i}{\sqrt{2}}\right) = \frac{1}{2}(1-i)$$

$$|\langle +\hat{x}|-\hat{y}\rangle|^2 = \frac{1}{4}(1+i)(1-i) = \frac{1}{2} \Rightarrow \text{Prob}(S_x = +\hbar/2) = 1/2$$

Similarly $\langle +\hat{x}|-\hat{y}\rangle = -\frac{1}{2}(1+i)$ $\Rightarrow \text{Prob}(S_x = -\hbar/2) = 1/2$

So we have seen how to use kets to compute probabilities of any measurement outcome on any input state of the form $|+\hat{m}\rangle$. What about more general states $|\Psi\rangle$?

We can show that for an arbitrary state

$$|\Psi\rangle = c_+ |+\hat{z}\rangle + c_- |-\hat{z}\rangle$$

which is normalized, there exist θ, ϕ and φ so that see pg 35-36

$$|\Psi\rangle = e^{i\varphi} \left\{ \cos(\theta/2) |+\hat{z}\rangle + e^{i\phi} \sin(\theta/2) |-\hat{z}\rangle \right\}$$

The term $e^{i\varphi}$ is a global phase that we can ignore. Thus the ket is equivalent to some state $|+\hat{n}\rangle$ where θ, ϕ are the spherical co-ordinates for $+\hat{n}$. Thus:

For any state $|\Psi\rangle$, there exists some direction $+\hat{n}$ so that $SG\hat{n}$ yields one particular outcome with certainty