Tues: HW by 5pm
Read ~~B~~
R. 2.3, 2.5
My notes pg 30-37, 40-50

Quantum states for spin-1/2 particles.

Experimental observations resulting from Stern-Gerlach experiments yield the following for spin-1/2 particles.

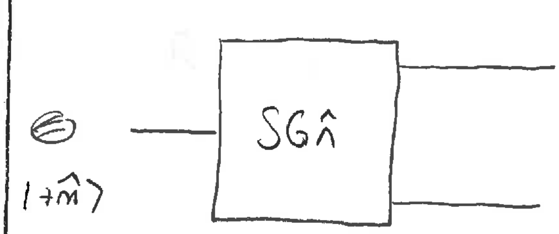
> For any unit vector $\hat{n}$ there is ~~are~~ two states of the particle that have the following notation + meaning:
>
> $|+\hat{n}\rangle \iff$ SG$\hat{n}$ measurement gives $S_n = +\hbar/2$ with certainty
>
> $|-\hat{n}\rangle \iff$ SG$\hat{n}$ " " $S_n = -\hbar/2$ " "

The repeatibility requirement for any single measurement means that:

> If SG$\hat{n}$ yields $+\hbar/2$ then, regardless of the state of the particle before measurement, its state after is $|+\hat{n}\rangle$.
> If it yields $-\hbar/2$ then its state after measurement is $|-\hat{n}\rangle$

Finally we can consider measurements along directions that are not compatible with input states:

If $\hat{m}, \hat{n}$ are unit vectors then



outcome	probability	state after
$S_n = +\hbar/2$	$\frac{1+\hat{m}\cdot\hat{n}}{2}$	$\lvert +\hat{n}\rangle$
$S_n = -\hbar/2$	$\frac{1-\hat{m}\cdot\hat{n}}{2}$	$\lvert -\hat{n}\rangle$

Mathematics for Spin-½ systems

So far we have a collection of states with clear meaning in terms of measurements:

$$\{ \, |+\hat{n}\rangle \; | \; \text{all unit vectors } \hat{n} \, \}$$ "kets"

We want to establish a mathematics for this that enables us to calculate probabilities, states, etc, ... just using these objects. The relevant mathematics will be that of vector spaces with complex number coefficients. We will

- show how to represent states via vectors
- " " " relate different states e.g. $|+\hat{x}\rangle = ?? \, |+\hat{z}\rangle \, ??|-\hat{z}\rangle$
- " " " compute probabilities via vectors.

Complex vector spaces.

The particularly useful vectors here will be two dimensional complex vectors. For example one such vector may be

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where u_1, u_2 are complex numbers. We can add + multiply these via:

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

vector addition

$$u + v := \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

multiplication by a scalar $\alpha \in \mathbb{C}$

$$\alpha u := \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix}$$

These are again two column vectors with complex entries. We can easily show that: For any such vectors u, v and scalars α, β.

i) $u + v = v + u$

ii) $(u+v)+z = u+(v+z)$

iii) $u + 0 = u$ where $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

iv) $\alpha(\beta u) = (\alpha\beta) u$

v) $(\alpha+\beta) u = \alpha u + \beta u$

vi) $\alpha(u+v) = \alpha u + \alpha v$

Note that if we define the special vectors

$$e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then any vector u has form

$$u = u_1 e_1 + u_2 e_2$$

where u_1, u_2 are complex numbers. There is a unique choice of two complex numbers for any vector. Summarizing:

> There exist two basis vectors $\{e_1, e_2\}$ so that any vector u can be expressed as
>
> $$u = u_1 e_1 + u_2 e_2$$
>
> where u_1, u_2 are two complex numbers, called the components of u in the basis $\{e_1, e_2\}$.

For any two dimensional vector space there are infinitely many choices of basis sets. The same vector will be represented by different components in different bases.

1 Vector bases

Consider vectors in two dimensions and the basis

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Suppose that the components of a vector $\mathbf{u}$ in this basis are $u_1 = 3$ and $u_2 = 4i$.

a) Consider an alternative basis

$$\mathbf{f}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{f}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Determine the components of $\mathbf{u}$ in this basis.

b) Consider an alternative basis

$$\mathbf{f}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{f}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Determine the components of $\mathbf{u}$ in this basis.

Answer: a) $u = 3e_1 + 4i\, e_2$

Then $f_1 + f_2 = \sqrt{2}\, e_1 \Rightarrow e_1 = \frac{1}{\sqrt{2}}(f_1 + f_2)$

$f_1 - f_2 = \sqrt{2}\, e_2 \qquad e_2 = \frac{1}{\sqrt{2}}(f_1 - f_2)$

So $u = \frac{3}{\sqrt{2}}(f_1 + f_2) + \frac{4i}{\sqrt{2}}(f_1 - f_2)$

$$= \underbrace{\frac{3+4i}{\sqrt{2}}}_{u_1} f_1 + \underbrace{\frac{3-4i}{\sqrt{2}}}_{u_2} f_2$$

b) $f_1 = 2e_1 \Rightarrow e_1 = \frac{1}{2} f_1 \qquad e_1 = \frac{1}{2} f_1$

$f_2 = e_1 + e_2 \Rightarrow f_2 = \frac{1}{2} f_1 + e_2 \Rightarrow e_2 = f_2 - \frac{1}{2} f_1$

So $u = \frac{3}{2} f_1 + 4i\left(f_2 - \frac{1}{2} f_1\right) = \underbrace{\left(\frac{3-4i}{2}\right)}_{u_1} f_1 + \underbrace{-2i}_{u_2} f_2$

We need another tool for computation that takes two vectors and produces a scalar. This is the inner product, which is a generalization of the dot product

Two vectors u, v $\longrightarrow$ scalar (complex number) (u,v) OR $\langle u,v \rangle$

One example would be as follows:

Given a particular basis e_1, e_2 and

$$u = u_1 e_1 + u_2 e_2 \sim \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$v = v_1 e_1 + v_2 e_2 \sim \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

then let

$$(u,v) := u_1^* v_1 + u_2^* v_2$$

This satisfies:

i) $(u, \alpha v + \beta z) = \alpha (u,v) + \beta (u,z)$

ii) $(v,u) = (u,v)^*$

iii) $(u,u) \geq 0$ with equality $\Leftrightarrow u = 0$

Given an inner product, we adopt the language:

Two vectors are orthogonal $\Leftrightarrow$ $(u,v) = 0$

A single vector is normalized $\Leftrightarrow$ $(u,u) = 1$

A set of basis vectors that are orthogonal + normalized is called an orthonormal basis

2 Vector bases

Consider the following vectors in two dimensions

$$\mathbf{f} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{g} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

a) Determine the inner product of each vector with itself, i.e. $(\mathbf{f}, \mathbf{f})$.

b) Determine the inner product $(\mathbf{f}, \mathbf{g})$.

c) Do these constitute an orthonormal basis?

Answer: a) $f_1 = i/\sqrt{2}$ $\quad f_2 = 1/\sqrt{2}$

So $(f, f) = f_1^* f_1 + f_2^* f_2$

$$= -\frac{i}{\sqrt{2}}\frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1$$

Then $g_1 = -i/\sqrt{2}$ $\quad g_2 = 1/\sqrt{2}$

So $(g, g) = g_1^* g_1 + g_2^* g_2$

$$= i/\sqrt{2}\left(\frac{-i}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1$$

b) $(f, g) = f_1^* g_1 + f_2^* g_2$

$$= \frac{-i}{\sqrt{2}}\left(\frac{-i}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 0$$

c) Yes, they are orthogonal and they are normalized.

Note that we can represent the inner product of two vectors as follows:

Two vectors

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Want (u, v)

Form a row vector from u with complex conjugate entries:

$$\begin{pmatrix} u_1^* & u_2^* \end{pmatrix}$$

Do matrix operation with row vector on column vector

$$\begin{pmatrix} u_1^* & u_2^* \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1^* v_1 + u_2^* v_2$$

This gives (u, v)

Ket vectors

The mathematical structure that we will use for the set of all kets is that the set of all ket vectors forms a complex vector space. This means that linear combinations of kets are again kets. For example:

Two kets:

$$|+\hat{z}\rangle \,,\; |+\hat{x}\rangle$$

$\longrightarrow$

The following are all kets:

$$3\,|+\hat{z}\rangle + 4\,|+\hat{x}\rangle$$

$$3\,|+\hat{z}\rangle - 4i\,|+\hat{x}\rangle$$

$$\alpha\,|+\hat{z}\rangle + \beta\,|+\hat{x}\rangle$$

where α, β are any numbers

Note that we still have to provide a label for such combinations:

$$\alpha\,|+\hat{z}\rangle + \beta\,|+\hat{x}\rangle = |\,\underbrace{\text{what label}}\,??\rangle$$

Is this $+\alpha\hat{z} + \beta\hat{x}$? No

We will eventually give such a rule. Note though that we will be able to do some mathematical operations. For example

$$\frac{1}{\sqrt{2}}\Big(|+\hat{z}\rangle + i\,|-\hat{z}\rangle\Big) + \Big(-i\,|+\hat{z}\rangle + -|\hat{z}\rangle\Big)\frac{1}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)|+\hat{z}\rangle + \left(\frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)|-\hat{z}\rangle$$

$$= \frac{1-i}{\sqrt{2}}\,|+\hat{z}\rangle + \frac{i-1}{\sqrt{2}}\,|-\hat{z}\rangle$$

At this stage we can just do such linear combinations symbolically. But we will need the notion of basis vectors to do various calculations. Based on the eventual rules for computing probabilities for SG measurements we will require:

Consider any SG$\hat{n}$ measurement. The states associated with the two mutually incompatible measurement outcomes for this measurement are orthogonal, and constitute a basis for the vector space of all kets

Thus $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$ constitute one basis. Similarly $\{|+\hat{x}\rangle, |-\hat{x}\rangle\}$ constitute a different basis. So we get that it must be possible to express:

$$|\text{any state}\rangle = c_+ |+\hat{z}\rangle + c_- |-\hat{z}\rangle$$

c_+ and c_- are complex coefficients. We often fix attention on the $(|+\hat{z}\rangle, |-\hat{z}\rangle)$ basis and write:

$$|+\hat{z}\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\hat{z}\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Another piece of notation is: The inner product of two states $|\Psi\rangle, |\Phi\rangle$ is written

$$(|\Phi\rangle, |\Psi\rangle) \equiv \langle\Phi|\Psi\rangle$$

So if $|\Psi\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle$

$|\Phi\rangle = b_+ |+\hat{z}\rangle + b_- |-\hat{z}\rangle$

then $\langle\Phi|\Psi\rangle = b_+^* a_+ + b_-^* a_-$

Note that

$$\langle +\hat{z}|+\hat{z}\rangle = 1$$

$$\langle -\hat{z}|-\hat{z}\rangle = 1$$

$$\langle +\hat{z}|-\hat{z}\rangle = 0$$

Finally note that

$$\langle\Phi|\Psi\rangle = \langle\Psi|\Phi\rangle^*$$

3 Ket algebra

Let

$$|\psi\rangle := 1\,|+\hat{\mathbf{z}}\rangle - 2i\,|-\hat{\mathbf{z}}\rangle \quad \text{and}$$
$$|\phi\rangle := \frac{1}{\sqrt{5}}\left(i\,|+\hat{\mathbf{z}}\rangle + 2i\,|-\hat{\mathbf{z}}\rangle\right).$$

a) Represent each ket as a column vector.

b) Determine whether each ket is normalized.

c) Determine $\langle\psi|\phi\rangle$.

d) Determine $\langle\phi|\psi\rangle$.

Answer: a) $|\psi\rangle \rightsquigarrow \begin{pmatrix} 1 \\ -2i \end{pmatrix}$ $\qquad |\phi\rangle \rightsquigarrow \begin{pmatrix} i/\sqrt{5} \\ 2i/\sqrt{5} \end{pmatrix}$

b) $\langle\psi|\psi\rangle = 1^* 1 + (-2i)^*(-2i) = 5$ no....

$\langle\phi|\phi\rangle = \left(+i/\sqrt{5}\right)^* \frac{i}{\sqrt{5}} + \left(\frac{2i}{\sqrt{5}}\right)^*\left(\frac{2i}{\sqrt{5}}\right) = \frac{5}{5} = 1$ yes

c) $$\langle\psi|\phi\rangle = \begin{pmatrix} 1 & +2i \end{pmatrix}\begin{pmatrix} i/\sqrt{5} \\ \frac{2i}{\sqrt{5}} \end{pmatrix} = \frac{i}{\sqrt{5}} + 2i\left(\frac{2i}{\sqrt{5}}\right)$$
$$= \frac{-4+i}{\sqrt{5}}$$

d) $$\langle\phi|\psi\rangle = \begin{pmatrix} \frac{-i}{\sqrt{5}} & \frac{-2i}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 1 \\ -2i \end{pmatrix} = \frac{-i}{\sqrt{5}} - \frac{2i}{\sqrt{5}}(-2i)$$
$$= \frac{-4-i}{\sqrt{5}}$$

Note $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$