Phys 396
Fall 2018

Quantum Information: Homework 7

Due: 16 October 2018

1 Unitary operators

Show that each of the following operators is unitary.

a) $\hat{\sigma}_z$
b) $\hat{\sigma}_y$
c) The Hadamard operator, $\hat{H}$.

2 Pauli operators

Pauli operators are important throughout the description of qubit operations. They have certain important algebraic properties.

a) Show that $\hat{\sigma}_x\hat{\sigma}_y = i\hat{\sigma}_z$.
b) Show that $\hat{\sigma}_y\hat{\sigma}_z = i\hat{\sigma}_x$.
c) Show that $\hat{\sigma}_z\hat{\sigma}_x = i\hat{\sigma}_y$.
d) Show that $\hat{\sigma}_z\hat{\sigma}_x = -\hat{\sigma}_x\hat{\sigma}_z$.
e) Show that $\hat{\sigma}_z\hat{\sigma}_y = -\hat{\sigma}_y\hat{\sigma}_z$.

In general for any of the standard unit vectors $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$,

$$\hat{\sigma}_m\hat{\sigma}_n = \hat{\mathbf{m}} \cdot \hat{\mathbf{n}}\,\hat{I} - i\hat{\sigma}_{m\times n}$$

where $m \times n$ represents $\hat{\mathbf{m}} \times \hat{\mathbf{n}}$.

3 Phase shift gate

Let

$$\hat{U} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

where φ is any real number. This is the phase shift gate.

a) Suppose that the state prior to the gate action is $|0\rangle$. Determine the state after the gate has acted on the qubit.
b) Suppose that the state prior to the gate action is $|1\rangle$. Determine the state after the gate has acted on the qubit.
c) Suppose that the state prior to the gate action is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Determine the state after the gate has acted on the qubit.

d) What value of φ maps

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)?$$

For this value of φ, determine the effect of the gate on $\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$.

e) What value of φ maps

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right)?$$

For this value of φ, determine the effect of the gate on $\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$

4 Rotation about the y axis

Consider the operator

$$\hat{U} = e^{-i\varphi\hat{\sigma}_y/2}$$

where φ is real.

a) Determine the matrix that represents $\hat{U}$.

b) Suppose that a single qubit is initially in the state, $|\Psi_0\rangle$ which corresponds to a Bloch sphere direction in the xz plane at an angle of θ from the z axis. Show that the state is represented by

$$|\Psi_0\rangle \leftrightsquigarrow \begin{pmatrix} \cos\left(\dfrac{\theta}{2}\right) \\ \sin\left(\dfrac{\theta}{2}\right) \end{pmatrix}.$$

c) Determine an expression for $\hat{U}\,|\Psi_0\rangle$ and show that it corresponds to a Bloch sphere direction that is rotated through angle φ about the $+y$ axis from the Bloch sphere direction corresponding to $|\Psi_0\rangle$.

5 Rotation about an arbitrary axis

This exercise considers the operator

$$\hat{U} = e^{-i\varphi(n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z)/2}$$

where $\hat{\mathbf{n}} = n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}}$ is any ordinary three dimensional unit vector. For convenience let

$$\hat{\sigma}_n := n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z$$

a) Show that $\hat{\sigma}_n^\dagger = \hat{\sigma}_n$ (i.e. $\hat{\sigma}_n$ is Hermitian).

b) Show that $\hat{\sigma}_n^2 = \hat{I}$. *Hint: Doing this algebraically using a result from an earlier problem is quicker than writing out matrices.*

c) Determine a matrix expression for $\hat{U}$.

d) Show that $\hat{U}$ is unitary.

Aside from an overall global phase this is the most general single qubit unitary. In terms of the Bloch sphere it can be show that it rotates the Bloch sphere vector representing the initial state through angle φ about the axis $\hat{\mathbf{n}}$.

6 News item

Find a news item published within the last year that describes an advance in quantum computing or quantum information. Post a link to the item in the discussion thread for 16 October. Summarize (here) in a single paragraph what the article describes.