Quantum Information: Homework 3

Due: 11 September 2018

1 Kets and Stern-Gerlach measurements.

a) A Stern-Gerlach apparatus is oriented in the direction $\hat{\mathbf{n}} = (\hat{\mathbf{x}} + \hat{\boldsymbol{z}})/\sqrt{2}$. Express, as a superposition of $|+\hat{\boldsymbol{z}}\rangle$ and $|-\hat{\boldsymbol{z}}\rangle$, the state which emerges from the measuring apparatus if it gave measurement outcome $S_n = +\hbar/2$. Repeat this for a particle emerging from the device if it yielded $S_n = -\hbar/2$. Demonstrate that the two states are orthogonal.

b) A Stern-Gerlach apparatus is oriented in the direction $\hat{\mathbf{n}} = (\hat{\mathbf{y}} + \hat{\boldsymbol{z}})/\sqrt{2}$. Express, as a superposition of $|+\hat{\boldsymbol{z}}\rangle$ and $|-\hat{\boldsymbol{z}}\rangle$, the state which emerges from the measuring apparatus if it gave measurement outcome $S_n = +\hbar/2$. Repeat this for a particle emerging from the device if it yielded $S_n = -\hbar/2$. Demonstrate that the two states are orthogonal.

2 Spin-1/2 states and Stern-Gerlach measurements

Consider a spin-1/2 system prepared in the state

$$|\Psi\rangle = A\big\{39\,|+\hat{\boldsymbol{z}}\rangle + (48+20i)\,|-\hat{\boldsymbol{z}}\rangle\big\}$$

where A is a normalization constant.

a) Apply the normalization condition to determine A.

b) Suppose that you would like to subject this particle to measurement via a Stern-Gerlach apparatus whose magnetic field is oriented in some direction $\hat{\mathbf{n}}$ so that the outcome of the measurement is $S_n = +\hbar/2$ with 100% certainty. Determine values for the spherical coordinate parameters θ, ϕ corresponding to $\hat{\mathbf{n}}$ which will ensure this.

3 Orthonormal kets and Stern-Gerlach measurements

Sometimes you will be given a complete description of the state of a spin-1/2 particle corresponding to the measurement outcome $S_n = +\hbar/2$ but will then have to deal with circumstances in which the particle is not in the state $|+\hat{\boldsymbol{n}}\rangle$ and the measurement actually returns $S_n = -\hbar/2$. Mathematically this corresponds to a situation where you have complete knowledge of $|+\hat{\boldsymbol{n}}\rangle$ but really need to work with $|-\hat{\boldsymbol{n}}\rangle$. The aim of this exercise is to learn how to translate between these.

First consider an example. A Stern-Gerlach apparatus is oriented in such a way that it produces $S_n = +\hbar/2$ for the state represented by

$$|+\hat{\boldsymbol{n}}\rangle = \frac{1}{\sqrt{2}}\,|+\hat{\boldsymbol{z}}\rangle + \frac{3+4i}{5\sqrt{2}}\,|-\hat{\boldsymbol{z}}\rangle$$

with certainty. In one particular run of the experiment on a particle in a different state, the measurement returns $S_n = -\hbar/2$. You need to determine an expression for the corresponding state, i.e. that which gives $S_n = -\hbar/2$ with certainty, so that you can predict what happens in subsequent SG measurements.

a) One way to determine an expression for this state is via the relevant spherical angles, θ and ϕ. Determine these for the state $|+\hat{\boldsymbol{n}}\rangle$ and use these to derive an expression for the state of the particle which gives $S_n = -\hbar/2$ with certainty. Verify that the two states are orthonormal.

Hopefully you noticed a pattern in the answers for this example. Now consider a general state

$$|\Psi_1\rangle = c_+ |+\hat{\boldsymbol{z}}\rangle + c_- |-\hat{\boldsymbol{z}}\rangle .$$

where $|c_+|^2 + |c_-|^2 = 1$.

b) Show that

$$|\Psi_2\rangle = c_-^* |+\hat{\boldsymbol{z}}\rangle - c_+^* |-\hat{\boldsymbol{z}}\rangle$$

is normalized and orthogonal to $|\Psi_1\rangle$. Describe why this (seemingly purely mathematical) result is relevant to the example of part a).

4 Bra vectors and inner products

Consider the kets

$$\begin{aligned} |\Phi_1\rangle &= \frac{5}{13} |+\hat{\boldsymbol{z}}\rangle - \frac{12}{13} |-\hat{\boldsymbol{z}}\rangle , \\ |\Phi_2\rangle &= \frac{3i}{5} |+\hat{\boldsymbol{z}}\rangle + \frac{4}{5} |-\hat{\boldsymbol{z}}\rangle , \text{and} \\ |\Phi_3\rangle &= \frac{1+i}{2} |+\hat{\boldsymbol{z}}\rangle + \frac{1-i}{2} |-\hat{\boldsymbol{z}}\rangle . \end{aligned}$$

a) For each $|\Phi_i\rangle$ determine an expression for the associated bra $\langle\Phi_i|$ in terms of $\langle+\hat{\boldsymbol{z}}|$ and $\langle-\hat{\boldsymbol{z}}|$.

b) Express each bra $\langle\Phi_i|$ as a row vector.

c) Use bra and ket operations to calculate each of $\langle\Phi_i|\Phi_j\rangle$.

5 Bra vectors and measurements

a) Suppose that a particle in the state

$$|+\hat{\boldsymbol{n}}\rangle = \frac{1}{\sqrt{2}} |+\hat{\boldsymbol{z}}\rangle + \frac{3-4i}{5\sqrt{2}} |-\hat{\boldsymbol{z}}\rangle$$

is subjected to an SG $\hat{\boldsymbol{y}}$ measurement. Use

$$|+\hat{\boldsymbol{y}}\rangle = \frac{1}{\sqrt{2}} |+\hat{\boldsymbol{z}}\rangle + i\frac{1}{\sqrt{2}} |-\hat{\boldsymbol{z}}\rangle$$

to compute $\langle+\hat{\mathbf{y}}|$ and use this to compute the probability with which $S_y = +\hbar/2$.

b) Suppose that $\hat{\mathbf{m}} = (\hat{\mathbf{y}} + \hat{\mathbf{z}})/\sqrt{2}$ and a particle emerges from an SG $\hat{\boldsymbol{m}}$ measurement with $S_m = +\hbar/2$. This particle is then subjected to an SG $\hat{\boldsymbol{n}}$ measuring device where $\hat{\mathbf{n}} = (\hat{\mathbf{x}} - \hat{\mathbf{y}})/\sqrt{2}$. Determine expressions for $\langle +\hat{\mathbf{n}}|$ and $\langle -\hat{\mathbf{n}}|$ and use these to compute the probabilities with which the outcomes $S_n = +\hbar/2$, and $S_n = -\hbar/2$ occur.

6 Measurement calculations using different orthonormal bases

Suppose that a spin-1/2 particle is prepared in the state

$$|\Psi\rangle = \frac{3+4i}{5\sqrt{2}} |+\hat{\boldsymbol{z}}\rangle + \frac{3-4i}{5\sqrt{2}} |-\hat{\boldsymbol{z}}\rangle$$

and is subsequently subjected to an SG $\hat{\boldsymbol{x}}$ measuring device. The aim of this exercise is to calculate the probabilities with which the two measurement outcomes occur in two ways: the first uses the $\{|+\hat{\boldsymbol{z}}\rangle, |-\hat{\boldsymbol{z}}\rangle\}$ basis and the second the $\{|+\hat{\boldsymbol{x}}\rangle, |-\hat{\boldsymbol{x}}\rangle\}$ basis.

a) Express the states corresponding to the two definite outcomes $S_x = +\hbar/2$ and $S_x = -\hbar/2$ in terms of the $\{|+\hat{\boldsymbol{z}}\rangle, |-\hat{\boldsymbol{z}}\rangle\}$ basis and use this to compute the probabilities with which $S_x = +\hbar/2$, and $S_x = -\hbar/2$ occur.

b) Express $|\Psi\rangle$ in terms of the $\{|+\hat{\boldsymbol{x}}\rangle, |-\hat{\boldsymbol{x}}\rangle\}$ basis and use this to compute the probabilities with which $S_x = +\hbar/2$, and $S_x = -\hbar/2$ occur. Do the results agree with the previous part?

7 Reiffel, *Quantum Computing*, 2.6, page 27.

The notation for states here is different to the usual notation for spin-1/2 states. These can be translated back and forth using:

$$\begin{aligned}
|0\rangle &\leftrightsquigarrow |+\hat{\boldsymbol{z}}\rangle, \\
|1\rangle &\leftrightsquigarrow |-\hat{\boldsymbol{z}}\rangle, \\
|+\rangle &\leftrightsquigarrow |+\hat{\boldsymbol{x}}\rangle, \\
|-\rangle &\leftrightsquigarrow |-\hat{\boldsymbol{x}}\rangle, \\
|i\rangle &\leftrightsquigarrow |+\hat{\boldsymbol{y}}\rangle, \text{ and} \\
|-i\rangle &\leftrightsquigarrow |-\hat{\boldsymbol{y}}\rangle.
\end{aligned}$$

You could use this translation. However, you could also use the fact that $|0\rangle$ and $|1\rangle$ are orthonormal and that

$$\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle, \\
|-\rangle &= \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle, \\
|i\rangle &= \frac{1}{\sqrt{2}} |0\rangle + \frac{i}{\sqrt{2}} |1\rangle, \text{ and} \\
|-i\rangle &= \frac{1}{\sqrt{2}} |0\rangle - \frac{i}{\sqrt{2}} |1\rangle.
\end{aligned}$$

Finally specifying a measurement basis means that one specifies the states that would give each of the outcomes with certainty. Thus measuring in the basis $\{|+\hat{\boldsymbol{x}}\rangle, |-\hat{\boldsymbol{x}}\rangle\}$ means that one does the measurement where one outcome is attained with certainty if the input state is $|+\hat{\boldsymbol{x}}\rangle$ and the other is attained with certainty if the input state is $|-\hat{\boldsymbol{x}}\rangle$; in this case, that would be an SG $\hat{\boldsymbol{x}}$ measurement.

8 News item

Find a news item published within the last year that describes an advance in quantum computing or quantum information. Post a link to the item in the discussion thread for 4 September. Summarize (here) in a single paragraph what the article describes.