Quantum Information: Class Exam II

13 November 2018

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Instructions

- There are 5 questions on 8 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Question 1

Consider the evolution described by

$$\hat{U} = e^{-i\hat{\sigma}_y \varphi/2}$$

a) Determine the matrix that represents $\hat{U}$ (in the standard $\{|0\rangle, |1\rangle\}$ basis).

$$e^{-i\sigma_y \varphi/2} = \hat{I} + \left(\frac{-i\hat{\sigma}_y \varphi}{2}\right) + \frac{1}{2!}\left(\frac{-i\hat{\sigma}_y \varphi}{2}\right)^2 + \frac{1}{3!}\left(\frac{-i\hat{\sigma}_y \varphi}{2}\right)^3 + \cdots$$

But $\hat{\sigma}_y^2 = \hat{I}$ and this gives

$$e^{-i\hat{\sigma}_y \varphi/2} = \hat{I} + \left(\frac{-i\varphi}{2}\right)\hat{\sigma}_y + \frac{1}{2!}\left(\frac{-i\varphi}{2}\right)^2 \hat{I} + \frac{1}{3!}\left(\frac{-i\varphi}{2}\right)^3 \hat{\sigma}_y + \cdots$$

Question 1 continued ...

$$= \hat{I}\left(1 - \frac{1}{2!}\left(\frac{\varphi}{2}\right)^2 + \dots\right) \quad - i\hat{\sigma}_y\left(\left(\frac{\varphi}{2}\right) - \frac{1}{3!}\left(\frac{\varphi}{2}\right)^3 + \dots\right)$$

$$= \hat{I}\cos(\varphi/2) - i\hat{\sigma}_y \sin(\varphi/2)$$

$$= \cos\varphi/2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sin\varphi/2$$

$$\hat{U} = \begin{pmatrix} \cos\varphi/2 & -\sin\varphi/2 \\ \sin\varphi/2 & \cos\varphi/2 \end{pmatrix}$$

b) Suppose that immediately prior to evolution the system is in the state $|0\rangle$. Determine φ so that immediately after the evolution, the system will be in the state $|1\rangle$.

$$|\psi_i\rangle = |0\rangle \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{so}$$

$$|\psi_F\rangle = \hat{U}|0\rangle \rightsquigarrow \begin{pmatrix} \cos\varphi/2 & -\sin\varphi/2 \\ \sin\varphi/2 & \cos\varphi/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi/2 \\ \sin\varphi/2 \end{pmatrix}$$

We need $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \boxed{\varphi = \pi}$

$1\cancel{2}\ 14$

Question 2

Verify that

$$\hat{U} := |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_y$$

is unitary.

Need to show $\hat{U}^+\hat{U} = \hat{I}$ Then

$$\hat{U}^+ = \{ |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_y \}^+$$

$$= (|0\rangle\langle 0|)^+ \otimes \hat{I}^+ + (|1\rangle\langle 1|)^+ \otimes \hat{\sigma}_y^+$$

$$= |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_y$$

So

$$\hat{U}^+\hat{U} = \{ |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_y \} \{ |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_y \}$$

$$= |0\rangle\cancelto{1}{\langle 0|0\rangle}\langle 0| \otimes \hat{I}\hat{I}$$

$$+ |0\rangle\cancelto{0}{\langle 0|1\rangle}\langle 1| \otimes \hat{I}\hat{\sigma}_y$$

$$+ |1\rangle\cancelto{0}{\langle 1|0\rangle}\langle 0| \otimes \hat{\sigma}_y\hat{I}$$

$$+ |1\rangle\underbrace{\langle 1|1\rangle}_{1}\langle 1| \otimes \underbrace{\hat{\sigma}_y\hat{\sigma}_y}_{\hat{I}}$$

$$= |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{I} = \hat{I} \otimes \hat{I} = \hat{I}$$

So it is unitary

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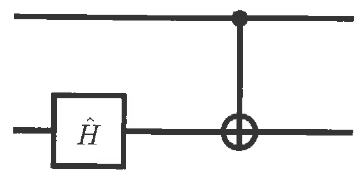
Question 3

Consider the illustrated circuit where $\hat{H}$ is a Hadamard gate.



a) Explain whether it is possible for this circuit to act on a system whose state is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

No, the given state has only one qubit and the circuit requires two

b) Determine the effect of this circuit on each of the four computational basis states: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

$$|0\rangle|0\rangle \xrightarrow{\hat{I}\otimes\hat{H}} |0\rangle(\hat{H}|0\rangle) = |0\rangle\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle+|01\rangle)$$

$$\xrightarrow{CNOT} = \frac{1}{\sqrt{2}}(|00\rangle+|01\rangle)$$

$$|0\rangle|1\rangle \xrightarrow{\hat{I}\otimes\hat{H}} |0\rangle\hat{H}|1\rangle = |0\rangle\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle-|01\rangle)$$

$$\xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|00\rangle-|01\rangle)$$

$$|1\rangle|0\rangle \xrightarrow{I\otimes\hat{H}} |1\rangle\hat{H}|0\rangle = |1\rangle\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|10\rangle+|11\rangle)$$

$$\xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|11\rangle+|10\rangle)$$

Question 3 continued . . .

$$|1\rangle|1\rangle \xrightarrow{\hat{H}} |1\rangle \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|10\rangle - |11\rangle\right)$$

$$\xrightarrow{CNOT} \frac{1}{\sqrt{2}}\left(|11\rangle - |10\rangle\right)$$

So

$$|00\rangle \to \frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right)$$
$$|01\rangle \to \frac{1}{\sqrt{2}}\left(|00\rangle - |01\rangle\right)$$
$$|10\rangle \to \frac{1}{\sqrt{2}}\left(|11\rangle + |10\rangle\right)$$
$$|11\rangle \to \frac{1}{\sqrt{2}}\left(|11\rangle - |10\rangle\right)$$

c) Suppose that the input state is

$$|\Psi\rangle := \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle).$$

Determine the output state produced by this circuit and verify whether it is entangled or not.

$$|\Psi\rangle \xrightarrow{\hat{I}\otimes\hat{H}} \frac{1}{2}\hat{I}\left(|0\rangle + |1\rangle\right)\hat{H}\left(|0\rangle - |1\rangle\right)$$

$$= \frac{1}{2}\left(|0\rangle + |1\rangle\right)\sqrt{2}|1\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right)$$

$$\xrightarrow{CNOT} \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle.$$

For product need $\underbrace{\alpha_0\alpha_3}_{0} = \underbrace{\alpha_1\alpha_2}_{1/2}$ not true $\Rightarrow$ <u>entangled</u>

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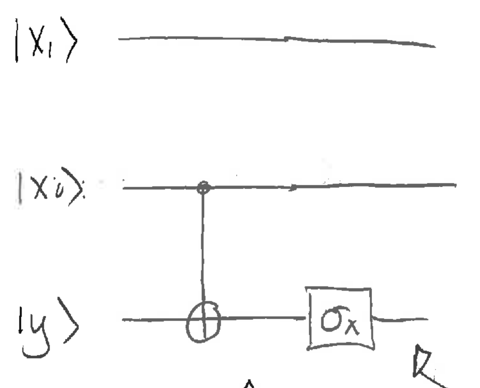
Question 4

Consider a function that maps two bits to a single bit according to $f(x_1, x_0) = x_0 \oplus 1$. An oracle that corresponds to this function maps three qubits to three qubits and is defined via the computational basis map

$$\hat{U}_f \left|x_1 x_0\right\rangle \left|y\right\rangle := \left|x_1 x_0\right\rangle \left|y \oplus f(x_1, x_0)\right\rangle$$

where $x_0, x_1, y = 0, 1$. Provide a three qubit circuit for this oracle using CNOT and single qubit gates.



here $|x_1 x_0\rangle | x_0 \oplus y\rangle$

here $|x_1 x_0\rangle \,|\, 1 \oplus x_0 \oplus y\rangle$

$= |x_1 x_0\rangle | y \oplus (x_0 + 1)\rangle$

$= |x_1 x_0\rangle | y \oplus f(x_1, x_0)\rangle$

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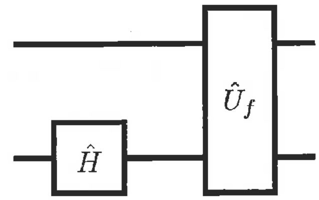
Question 5

Let f be a function that maps a single bit to a single bit and let $\hat{U}_f$ be the oracle defined on single qubit computational basis states as

$$\hat{U}_f |x\rangle |y\rangle := |x\rangle |y \oplus f(x)\rangle .$$

The lower (function register) qubit is initially in the state $|0\rangle$.



a) Suppose that f is constant. Determine the state after the oracle for any $x = 0$ or 1. Does this state depend on the particular constant function?

Initially $\quad |x\rangle|0\rangle \xrightarrow{I \otimes \hat{H}} |x\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$= \frac{1}{\sqrt{2}}\left(|x\rangle|0\rangle + |x\rangle|1\rangle\right)$$

$$\xrightarrow{\hat{U}_F} \frac{1}{\sqrt{2}}\left(|x\rangle|0 \oplus f(x)\rangle + |x\rangle|1 \oplus f(x)\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|x\rangle|f(x)\rangle + |x\rangle|1 \oplus f(x)\rangle\right)$$

So the final state is

$$|x\rangle \frac{1}{\sqrt{2}}\left(|f(x)\rangle + |1 \oplus f(x)\rangle\right)$$

Suppose f is constant: $f = 0$ then the function register gives $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. If $f = 1$ then the function register gives $\frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

Thus for constant functions we get

$$|x\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

The state does <u>not</u> depend on the function

Question 5 continued ...

b) Suppose that $f(x) = x$. Is the state of the system after the oracle any different to that for constant f? Explain your answer.

Then the state is

$$|x\rangle \frac{1}{\sqrt{2}}\left(|x\rangle + |1 \oplus x\rangle\right)$$

If $x=0$: then the function register gives $\left(|0\rangle + |1\rangle\right)\frac{1}{\sqrt{2}}$

If $x=1$ " " " " " $\left(|1\rangle + |0\rangle\right)\frac{1}{\sqrt{2}}$

So regardless of x the function register is $\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$.

Thus state is:

$$|x\rangle \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

This is the <u>same</u> for a constant function

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