Phys 396
Fall 2018

Quantum Information: Class Exam 1

9 October 2018

Name: ______________________________ **Total:** /~~50~~ 70

Instructions

- There are 6 questions on 8 pages.
- Show your reasoning and calculations and always explain your answers.

Question 1

Consider two binary digits, a and b. Let the NAND of these be denoted $\mathsf{NAND}(a,b)$. Show that for all a and b,

$$\mathsf{NAND}(a,b) = 1 \oplus (ab).$$

a	b	NAND	ab	$1 \oplus ab$
0	0	1	0	1
0	1	1	0	1
1	0	1	0	1
1	1	0	1	$1 \oplus 1 = 0$

these match

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Question 2

A spin-1/2 particle is subjected to a Stern-Gerlach measurement where the apparatus measures the spin component along the direction $\hat{\mathbf{n}} = \frac{\sqrt{3}}{2}\,\hat{\mathbf{y}} + \frac{1}{2}\,\hat{\mathbf{z}}$. The outcome of the measurement is $S_n = +\hbar/2$

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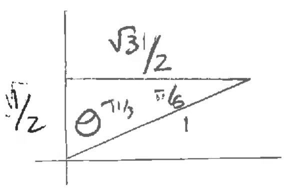
a) Express the state of the particle after this measurement in the basis $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.

Here $\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \tan^{-1}(\sqrt{3}) = 60^\circ = \pi/3$



$\phi = \pi/2$

It emerges in state $|+\hat{n}\rangle$ so

$$|+\hat{n}\rangle = \cos\theta/2\, |+\hat{z}\rangle + e^{i\phi} \sin\theta/2\, |-\hat{z}\rangle$$

$$= \cos(\pi/6)\, |+\hat{z}\rangle + e^{i\pi/2} \sin\pi/6\, |-\hat{z}\rangle$$

$$= 1/2\, |+\hat{z}\rangle + i\frac{\sqrt{3}}{2}\, |-\hat{z}\rangle$$

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b) Immediately after this measurement the particle is subjected Stern-Gerlach measurement where the apparatus measures the spin component along the direction $\hat{\mathbf{y}}$. List the possible outcomes of this measurement and the probabilities with which they occur.

S_y	Associated State	Prob
$+\hbar/2$	$\vert+\hat{y}\rangle$	$\vert\langle+\hat{y}\vert+\hat{n}\rangle\vert^2 = \frac{2+\sqrt{3}}{4}$
$-\hbar/2$	$\vert-\hat{y}\rangle$	$\vert\langle-\hat{y}\vert+\hat{n}\rangle\vert^2 = \frac{2-\sqrt{3}}{4}$

Now $|+\hat{y}\rangle = \frac{1}{\sqrt{2}}\left(|+\hat{z}\rangle + i\,|-\hat{z}\rangle\right)$

$$\langle+\hat{y}|+\hat{n}\rangle = \frac{1}{\sqrt{2}}\left(\langle+\hat{z}| - i\langle-\hat{z}|\right)\left(\frac{1}{2}|+\hat{z}\rangle + i\frac{\sqrt{3}}{2}|-\hat{z}\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{2} - \frac{i\,i\sqrt{3}}{\sqrt{2}\;2} = \frac{1+\sqrt{3}}{2\sqrt{2}}$$

Question 2 continued . . .

$$|\langle +\hat{y}|+\hat{n}\rangle|^2 = \frac{(1+\sqrt{3})^2}{4\cdot 2} = \frac{4+2\sqrt{3}}{8} = \frac{2+\sqrt{3}}{4}.$$

Now $|-\hat{y}\rangle = 1/\sqrt{2}\ |+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle$

$$\langle -\hat{y}|+\hat{n}\rangle = \frac{1}{\sqrt{2}}\left(\langle +\hat{z}| + \frac{i}{\sqrt{2}}\langle -\hat{z}|\right)\left(\frac{1}{2}|+\hat{z}\rangle + \frac{i\sqrt{3}}{2}|-\hat{z}\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}(1-\sqrt{3})$$

So $|\langle -\hat{y}|+\hat{n}\rangle|^2 = \frac{2-\sqrt{3}}{4}$

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4 c) Immediately after the SG $\hat{y}$ measurement the particle is subjected Stern-Gerlach measurement where the apparatus measures the spin component along the original direction $\hat{\mathbf{n}}$. Will the outcome $S_n = +\hbar/2$ occur again with certainty? Explain your answer.

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No. For example, if one obtained $S_y = +\hbar/2$, then

$$\text{Prob}(S_n = +\hbar/2) = |\langle +\hat{n}|+\hat{y}\rangle|^2 = |\langle +\hat{y}|+\hat{n}\rangle|^2 = \frac{2+\sqrt{3}}{4}$$

If one obtained $S_y = -\hbar/2$ then

$$\text{Prob}(S_n = +\hbar/2) = |\langle +\hat{n}|-\hat{y}\rangle|^2 = \frac{2-\sqrt{3}}{4}$$

These do not add to 1. So $S_n = +\hbar/2$ does not occur with certainty.

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Question 3 Do either a or b

a) Consider the following polarization state of a single photon:

$$|\psi\rangle = \frac{1}{5}\Big(3i\,|{\rightarrow}\rangle + 4\,|{\uparrow}\rangle\Big).$$

Determine an expression, in terms of $\Big\{|{\rightarrow}\rangle\,,|{\uparrow}\rangle\Big\}$, for a normalized state that is orthogonal to $|\psi\rangle$.

Need $|\phi\rangle = \alpha|{\rightarrow}\rangle + \beta\,|{\uparrow}\rangle$, s.t

$\langle\phi|\psi\rangle = 0 \quad \Rightarrow \quad \alpha\,\frac{3i}{5} + \beta\,\frac{4}{5} = 0$

$\Rightarrow \quad \alpha\, 3i + \beta 4 = 0$

$\Rightarrow \quad \beta = -\frac{3i}{4}\alpha$

So $\alpha = 4 \quad \beta = -3i$ satisfies this. To normalize

$|\alpha|^2 + |\beta|^2 = 5^2 \quad \Rightarrow$

$\alpha = \frac{4}{5} \quad \beta = -\frac{3i}{5}$

$|\phi\rangle = \frac{1}{5}\left(4|{\rightarrow}\rangle - 3i\,|{\uparrow}\rangle\right)$

works

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Question 4

b) Two possible single qubit measurements are proposed. For each measurement the two associated states are listed as follows.

Possible measurement A: $\left\{ \overbrace{\frac{1}{\sqrt{5}}\big(|0\rangle - 2\,|1\rangle\big)}^{|\phi_1\rangle}, \overbrace{\frac{1}{\sqrt{5}}\,(2\,|0\rangle + |1\rangle)}^{|\phi_2\rangle} \right\}$

Possible measurement B: $\left\{ \underbrace{\frac{1}{\sqrt{5}}\big(|0\rangle - 2i\,|1\rangle\big)}_{|\psi_1\rangle}, \underbrace{\frac{1}{\sqrt{5}}\,(|0\rangle + 2i\,|1\rangle)}_{|\psi_2\rangle} \right\}$

Determine whether each of these could represent a measurement. Explain your answer.

The states need to be orthogonal. Try A:

$$\langle \phi_1 | \phi_2 \rangle = \frac{1}{\sqrt{5}}(\langle 0| - 2\langle 1|)\frac{1}{\sqrt{5}}(2|0\rangle + |1\rangle) = \frac{1}{5}(2-2) = 0$$

Each is normalized $\Rightarrow$ A is a possible measurement

Try B:

$$\langle \psi_1 | \psi_2 \rangle = \frac{1}{\sqrt{5}}(\langle 0| + 2i\langle 1|)\frac{1}{\sqrt{5}}(|0\rangle + 2i|1\rangle)$$

$$= \frac{1}{5}(1-4) = -3/5 \neq 0$$

These are not orthogonal. This is not a measurement

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Question ~~5~~ 4

Two qubits are in the state

$$|\Psi\rangle\ \cancel{|\psi\rangle} = \frac{1}{2}\Big(|00\rangle - |01\rangle - |10\rangle + i\,|11\rangle\Big).$$

a) Determine whether this is an entangled state or a product state.

If $|\psi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$ then this is entangled

$\Leftrightarrow \quad \alpha_0\alpha_3 = \alpha_1\alpha_2$

Here $\quad \alpha_0\alpha_3 = \frac{1}{2}\left(\frac{i}{2}\right) = \frac{i}{4}$

$\alpha_1\alpha_2 = -\frac{1}{2}\left(-\frac{1}{2}\right) = 1/4 \quad \neq \alpha_0\alpha_3 \quad$ entangled. 4

b) The "left" qubit is measured in the basis

$$\left\{\frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big), \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right\}$$

and the "right" qubit in the basis

$$\Big\{|0\rangle\,,|1\rangle\Big\}$$

Determine the probability with which all four measurement outcomes occur.

For left

outcome	state
+	$\frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)$
−	$\frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)$

For right

outcome	state
0	$\lvert 0\rangle$
1	$\lvert 1\rangle$

Question 5 continued ...

For both

Outcome		
Left	Right	Prob
+	0	0
+	1	1/4
−	0	1/2
−	1	1/4

$$\text{Prob}(+0) = \left| \tfrac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\langle 0| \, |\Psi\rangle \right|^2$$

$$= \left| \tfrac{1}{\sqrt{2}}(\langle 00| + \langle 10|)\, |\Psi\rangle \right|^2$$

$$= \frac{1}{2} \left| (\langle 00| + \langle 10|) \tfrac{1}{2}(|00\rangle - |01\rangle - |10\rangle + i|11\rangle) \right|^2$$

$$= \frac{1}{8} |1-1|^2 = 0$$

$$\text{Prob}(+1) = \left| \tfrac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\langle 1| \, |\Psi\rangle \right|^2 = \left| \tfrac{1}{\sqrt{2}}(\langle 01| + \langle 11|) \tfrac{1}{2}(|00\rangle - |01\rangle - |10\rangle + i|11\rangle) \right|^2$$

$$= \frac{1}{2}\frac{1}{4} |-1+i|^2 = \frac{1}{8}(1+1) = 1/4$$

$$\text{Prob}(-0) = \left| \tfrac{1}{\sqrt{2}}(\langle 0| - \langle 1|)\langle 0| \, |\Psi\rangle \right|^2$$

$$= \left| \tfrac{1}{\sqrt{2}}(\langle 00| - \langle 10|) \tfrac{1}{2}(|00\rangle - |01\rangle - |10\rangle + i|11\rangle) \right|^2 = \frac{1}{8}|2|^2 = 1/2$$

$$\text{Prob}(-1) = \left| \tfrac{1}{\sqrt{2}}(\langle 01| - \langle 11|)\, |\Psi\rangle \right|^2$$

$$= \left| \tfrac{1}{\sqrt{2}}(\langle 01| - \langle 11|) \tfrac{1}{2}(|00\rangle - |01\rangle - |10\rangle + i|11\rangle) \right|^2$$

$$= \left(\tfrac{1}{\sqrt{2}}\tfrac{1}{2}\right)^2 |-1-i|^2 = \frac{1}{4}$$

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Question ~~6~~ 5

Consider a single qubit measurement in the basis

$$\left\{\underbrace{\frac{1}{\sqrt{2}}\Big(|0\rangle + i\,|1\rangle\Big)}_{|\varphi_+\rangle}, \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle - i\,|1\rangle\right)}_{|\varphi_-\rangle}\right\}$$

Construct the two associated projector operators and, for each, verify that $\hat{P}^2 = \hat{P}$.

$$\hat{P}_+ = |\varphi_+\rangle\langle\varphi_+| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$$

$$\hat{P}_- = |\varphi_-\rangle\langle\varphi_-| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & +i\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$$

$$\hat{P}_+^2 = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}2 & -2i\\ 2i & 2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} = \hat{P}_+$$

$$\hat{P}_-^2 = \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}2 & +2i\\ -2i & 2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & +i\\ -i & 1\end{pmatrix} = \hat{P}_-$$

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