Laboratory 2: Numerical Computation of an Electric Field–Prelab

1 Field due to three point charges on a line

Three point particles, each with charge $+1.0 \times 10^{-9}\,\mathrm{C}$ lie along the x axis as illustrated. They are evenly spaced. Determine both components of the total electric field produced by these charges at point P as illustrated in Fig 1.

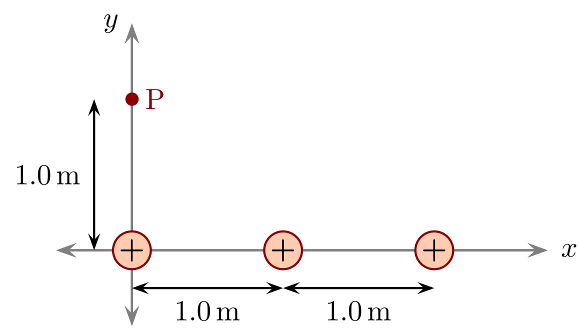

Figure 1: Three linear charges

Laboratory 2: Numerical Computation of an Electric Field–Activity

It is often necessary to determine the electric field produced by a continuous charge distribution. This laboratory describes a general procedure for doing this analytically (i.e. just using calculus and algebra). Sometimes it is possible to carry out the resulting mathematical manipulations. However, in most situations there is no known way to do this analytically. Fortunately, it is always possible to compute the electric fields to a high degree of accuracy numerically (i.e. by having a computer carry out a vast number of summations). Such **numerical approximations** are important in many sciences.

This laboratory employs both techniques to compute the electric field in a simple situation: that of a uniformly charged rod.

1 Electric Field Produced by a Charged Rod

The aim of this exercise will be to determine the electric field produced by a straight charged rod at any point perpendicular to one end of the rod; the scenario is illustrated in Fig. 2. We focus on the simplest situation, in which the rod carries charge Q, **uniformly distributed** along its length L.

In order to analyze this we set up axes as illustrated and aim to determine the electric field at the point P, with coordinates $(0, y)$.

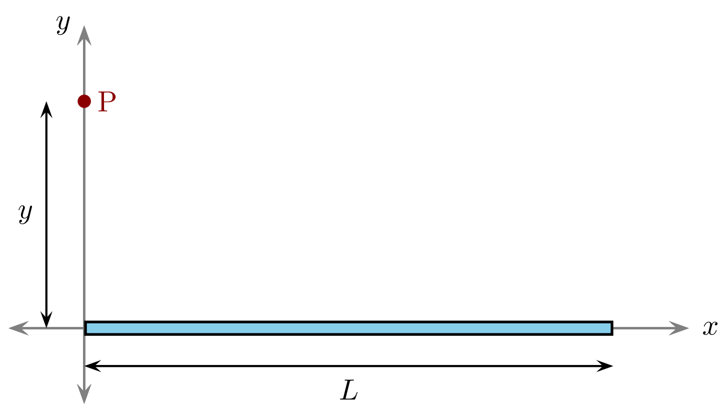

Figure 2: Charged rod and location of the field point, P.

The procedure for calculating the electric field in such cases involves splitting the rod in to small sections of equal length as illustrated in Fig. 3 and treating each as a point charge. The field produced by each segment can be approximated as a field produced by a point charge. The total electric field is computed by summing the electric fields produced by all the segments. The resulting electric field will be an approximation, whose accuracy improves when recalculated using smaller segments.

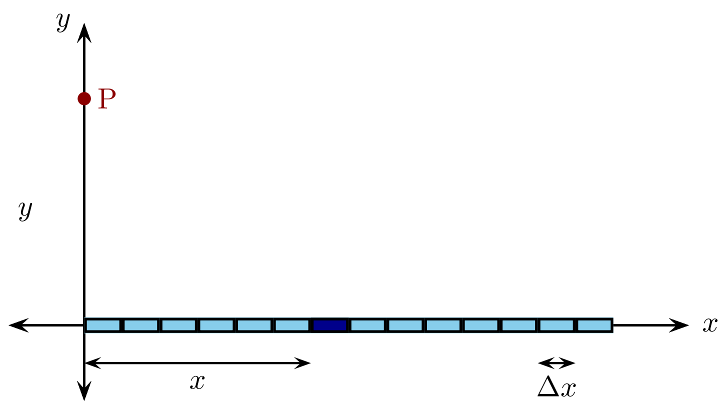

Figure 3: Decomposition of a continuous charge distribution into pointlike segments.

We start by determining the field produced by the segment whose left edge is located at x (see Fig. 3).

a) Determine an expression for the linear charge density, λ, in terms of Q and L.
b) Denote the charge contained in the shaded segment in Fig. 3 by Δq. Determine an expression, in terms of λ and Δx, for Δq.
c) Provide a diagram similar to that of Fig. 3 which shows the axes, the point at which the field is to be calculated, **only the shaded section of the charged rod** and the variables x, y and Δx appropriately. Indicate the electric field vector produced by the shaded segment at the location $(0, y)$ on this diagram.
d) Determine an expression for the **magnitude of the electric field,** denoted ΔE, produced by the shaded segment illustrated in Fig. 3. For simplicity, assume that the segment is equivalent to a point charge located at $(x, 0)$.
e) Determine an expression for the x component of the electric field, denoted ΔE_x, produced by the shaded segment illustrated in Fig. 3.
f) Determine an expression for the y component of the electric field, denoted ΔE_y, produced by the shaded segment illustrated in Fig. 3.
g) **Verify with the instructor** that your expressions for ΔE_x and ΔE_y are correct.
h) This process must be carried out for each segment of the charged rod. However, you have obtained expressions for ΔE_x and ΔE_y which will work for every segment. Identify the terms in each of these expressions which are the same for all segments and those which change from one segment to another.

The x and y-components of the electric field produced by the rod are

$$E_x = \sum_{\text{all segments}} \Delta E_x \tag{1}$$

$$E_y = \sum_{\text{all segments}} \Delta E_y. \tag{2}$$

i) Use Eqs. (1) and (2) to verify that

$$E_x = -\frac{1}{4\pi\epsilon_0}\frac{Q}{L}\sum_{\text{all segments}} \frac{x}{(x^2+y^2)^{3/2}}\,\Delta x \tag{3}$$

$$E_y = \frac{1}{4\pi\epsilon_0}\frac{Qy}{L}\sum_{\text{all segments}} \frac{1}{(x^2+y^2)^{3/2}}\,\Delta x. \tag{4}$$

Eqs. (3) and (4) are the main equations which will guide the computation of the electric field. It should be clear that the crux of this problem is to calculate the summations. The terms preceding these are constants that can be inserted later. Thus the main calculations are:

$$\sum_{\text{all segments}} \frac{x}{(x^2+y^2)^{3/2}}\,\Delta x \tag{5}$$

$$\sum_{\text{all segments}} \frac{1}{(x^2+y^2)^{3/2}}\,\Delta x. \tag{6}$$

2 Numerical Computation

In the following consider a rod of length 1.0 m.

a) In order to calculate the electric field components by hand consider the case where $y = 0.50\,\text{m}$ and there are four segments. Number the segments, $1, 2, 3, \ldots$ from left to right. Use a table of the following form to carry out, with a calculator, the summations of Eqs. (5) and (6).

Segment number	x	Δx	$x\Delta x/(x^2+y^2)^{3/2}$	$\Delta x/(x^2+y^2)^{3/2}$
1	0	0.25	0	2
2	...	...	...	...
3	...	...	...	...
4	...	...	...	...
Sum			...	...

Use the results to obtain an expressions for the x and y-components of the electric field.

b) This method uses approximations which become more accurate as the segment size decreases and the number of segments increases. The resulting summations can be computed efficiently by having a computer. Set up an Excel spreadsheet, formatted like the table above but which is configured to manage a large number of segments. This should be as automated as possible and there should be cells into which the number of segments and the value of y are entered. All calculations should refer to these cells when necessary. Verify that your spreadsheet reproduces the values that you computed in the previous part.

c) Carry out the calculation for $10, 50$ and 100 segments. For each case use the results to obtain an expressions for the x and y-components of the electric field.

d) To what values do the x and y-components of the electric field appear to converge as the number of segments increases?

3 Analytical Computation

In the limit as the number of segments approaches infinity, $\Delta x \to 0$ and the summations reduce to the following integrals.

$$\sum_{\text{all segments}} \frac{x}{(x^2+y^2)^{3/2}}\,\Delta x \to \int_0^L \frac{x}{(x^2+y^2)^{3/2}}\,\mathrm{d}x \tag{7}$$

$$\sum_{\text{all segments}} \frac{1}{(x^2+y^2)^{3/2}}\,\Delta x \to \int_0^L \frac{1}{(x^2+y^2)^{3/2}}\,\mathrm{d}x. \tag{8}$$

Standard techniques for evaluating these integrals give:

$$\int_0^L \frac{x}{(x^2+y^2)^{3/2}}\,\mathrm{d}x = -\frac{1}{\sqrt{x^2+y^2}}\Bigg|_0^L$$

$$\int_0^L \frac{1}{(x^2+y^2)^{3/2}}\,\mathrm{d}x = \frac{x}{y^2\sqrt{x^2+y^2}}\Bigg|_0^L$$

a) Use the exact integrals to determine an exact expressions for the E_x and E_y, each in terms of Q, L and y.

b) Evaluate the exact expressions for the E_x and E_y when $L = 1.0\,\mathrm{m}$ and $y = 0.50\,\mathrm{m}$ (the answer will still contain Q). Compare these to the results of your numerical simulation with 100 segments and $L = 1.0\,\mathrm{m}$ and $y = 0.50\,\mathrm{m}$.

c) Evaluate the exact expressions for the E_x and E_y when $L = 1.0\,\mathrm{m}$ and $y = 0.25\,\mathrm{m}$ (the answer will still contain Q). Compare these to the results of your numerical simulation with 100 segments and $L = 1.0\,\mathrm{m}$ and $y = 0.25\,\mathrm{m}$.

d) Suppose that $y \gg L$ (i.e. y is very much larger than L). Use the exact expressions for the E_x and E_y to determine approximate expressions for E_x and E_y and show that these resemble the results at a distance y from a point charge of magnitude Q. Discuss why you may have expected this.

e) This entire exercise has provided a rule for computing the electric field at any location above a rod of any length which is uniformly charged. Explain (without actually doing the calculation) how this rule could be used to compute the electric field at a point **anywhere above or below** the rod.