Atoms interacting with an electromagnetic field.

We have seen that an ensemble of two level atoms and a field that interact in a cavity are described by

$N =$ total number of atoms

$N_i =$ number of atoms in state i

$u(f) =$ spectral energy density of field.

Then, the atoms and the field will be in equilibrium when

$$u(f) = \frac{A_{21}}{B_{12}} \frac{1}{N_1/N_2 - 1} \quad \text{and} \quad \frac{N_1}{N_2} = e^{hf/k_B T}$$

where A_{21} is the spontaneous emission rate,

B_{12} is the absorption rate.

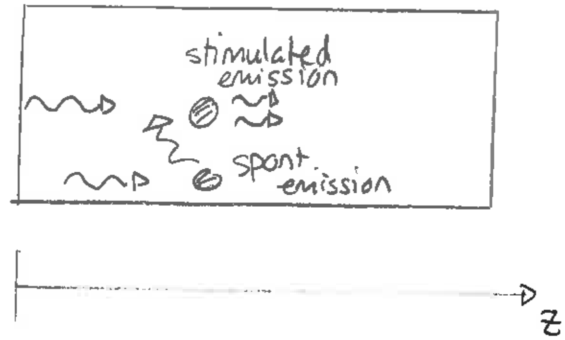
Denote these by A and B respectively. Thus

$$\boxed{u(f) = \frac{A}{B} \frac{1}{N_1/N_2 - 1}} \iff N_2 A = u B (N_1 - N_2)$$

Beam traversing a cavity

We now broaden this to consider a beam traversing a cavity that is populated with identical atoms. As the beam traverses the cavity, all three interactions can occur. We aim to determine which of these enhance and which reduce the beam intensity as it propagates.

The set up is as illustrated and we consider a beam initially localized at the left and propagating along $z>0$.



As the electromagnetic field encounters atom there are three processes:

1) absorption – removes energy from the beam.

2) stimulated emission – adds energy to the beam; this propagates along the beam direction.

3) spontaneous emission – adds electromagnetic energy but this propagates in random directions and gives negligible contributions to the beam energy. This energy is effectively lost from the system.

Suppose that the spontaneous emission rate were negligible. Then there are two competing processes. Intuitively if $N_2 > N_1$ then stimulated emission will dominate. If $N_2 < N_1$ then absorption will dominate. The beam will be enhanced or amplified whenever $N_2 > N_1$ and suffer a diminution whenever $N_2 < N_1$. We seek to make this precise, by determining a rule for

$I(z)$ = intensity of beam as a function of location along the cavity

To obtain a description of this, we need two results. First intensity is related to spectral energy density by

$$I(f) = c\, u(f)$$

if we have a monochromatic beam

In general beams are not monochromatic and this rule must be modified via a lineshape function. However, this is not important to the basic argument, and we consider a simplified situation where it can be omitted

Second, the Poynting theorem gives:

$$\boxed{\frac{\partial I}{\partial z} = -\frac{\partial u}{\partial t}}$$

$\Rightarrow$ energy stored in region

energy in field is opposite to energy in atoms.

<u>Proof</u>. Poynting's theorem states:

when energy in region accumulates then energy in > energy out.

$$\frac{\partial}{\partial t}\int_{\text{region}} u \, d\tau = -\oint_{\text{surface}} \vec{S}\cdot d\vec{a} = -\int_{\text{region}} \vec{\nabla}\cdot\vec{S}\, d\tau$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{S}$$

In this case S only varies along z. Thus

$$\frac{\partial u}{\partial t} = -\frac{\partial S}{\partial z}$$

But S gives the instantaneous intensity. Thus

$$\frac{\partial u}{\partial t} = -\frac{\partial I}{\partial z}$$

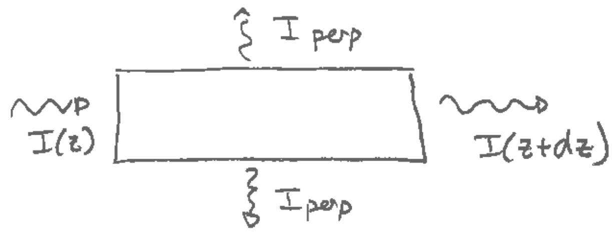
which gives the result $\blacksquare$

We now apply this by considering how energy may be lost from the beam in a given region.

First suppose that the beam is attenuated. Thus $\frac{\partial I}{\partial z} < 0$ and $\frac{\partial u}{\partial t} > 0$. This implies that energy accumulates in the cavity. But this cannot be so in a steady state. Energy must be lost.

This can only occur when one considers energy lost in directions perpendicular to the direction of propagation

Energy will be balanced if



$$\frac{\partial u}{\partial t} = \text{rate at which energy is emitted in perpendicular direction.}$$

The only way in which energy is lost in the perpendicular direction is via spontaneous emission. So

$$\frac{\partial u}{\partial t} = \underbrace{N_2 A}_{\text{rate at which photons are emitted spontaneous.}} \overbrace{hf}^{\text{energy per photon.}}$$

Thus

$$\frac{\partial I}{\partial z} = - N_2 A\, hf$$

But in equilibrium.

$$N_2 A = B(N_1 - N_2) u(f)$$

$$\Rightarrow \frac{\partial I}{\partial z} = - B(N_1 - N_2)\, hf\, u(f)$$

and $u(f) = I(f)/c$ implies:

$$\frac{\partial I}{\partial z} = \frac{B(N_2 - N_1)\, hf}{c} I$$

We see that in this simplied model that ignores linewidths and also spontaneous emission, the beam intensity varies exponentially along the cavity

$$\boxed{\begin{aligned} &I(z) = I(0)\, e^{G(z)} \\ &\text{where} \\ &G = B(N_2 - N_1)\tfrac{hf}{c} \end{aligned}}$$

Here G is the gain coefficient for this simple model.

Exercise: How must N_2 and N_1 be related for:

i) the beam to be attenuated along the cavity

ii) " " " " amplified " " "

and which of these will occur in thermal equilibrium?

Answer: If $G < 0$ then attenuation occurs

$\Rightarrow$ if $N_1 > N_2$ attenuation occurs.

Likewise if $N_2 > N_1$ amplification occurs.

In thermal equilibrium $N_1/N_2 = e^{hf/k_BT}$ which is always greater than 1.

Thus

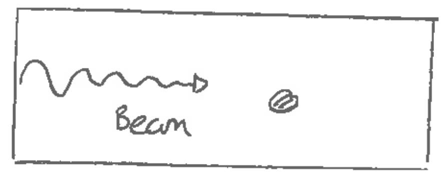
In thermal equilibrium the beam will be attenuated as it traverses the cavity

This is exacerbated by the presence of spontaneous emission. In practice there will be other factors causing losses. These include scattering of photons of atoms, resulting absorption by the cavity walls and so on. Additionally the beam can be made to traverse the cavity back + forth by reflecting mirrors placed at the cavity ends. Imperfections in these mirrors cause further losses.

Thermal equilib.

$N_1 > N_2$

$\Rightarrow$ absorption more likely than emission.

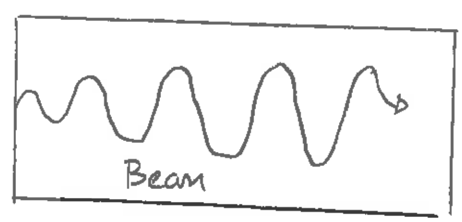


This attenuation is the norm. However if it were possible to invert the populations away from the thermal equilibrium requirement then $N_2 > N_1$ and the beam would be amplified.

Population inversion

$N_2 > N_1$

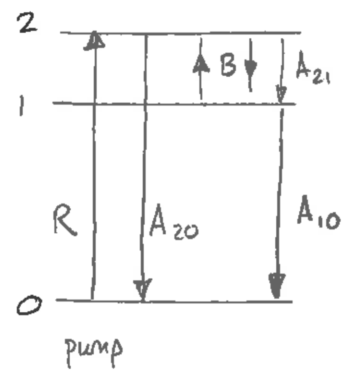
$\Rightarrow$ stimulated emission more likely than absorption



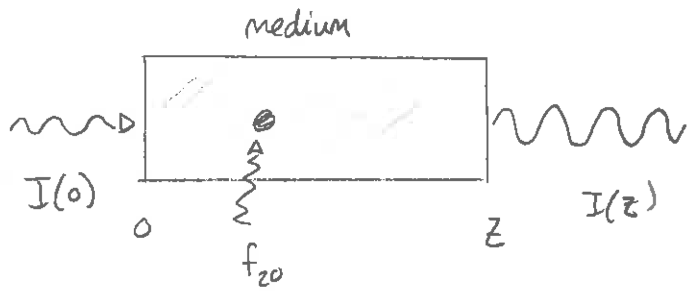
The issue is how to attain population inversion.

Population Inversion

We need a separate mechanism to generate a situation where the population of the higher state exceeds that of the lower state. This cannot be done with a two state atom alone as any pumping mechanism would require an electromagnetic field with the same frequency as that of the beam that is to be amplified. One model involves a three state system with energy levels that are not equally spaced. Suppose that we desire amplification of a wave with frequency



$$f_{21} = \frac{E_2 - E_1}{h}.$$



We can arrange for the system to be such that spontaneous emission from 0 to 1 is rapid. Then we can pump with an external beam tuned to the $0 \to 2$ transition (denote the frequency by f_{02}). There will be various absorption + emission processes with corresponding coefficients for the three levels. If the pump rate $R \ll A_{20}, A_{10}$ then we can show that

$$N_2 - N_1 \approx N \frac{R(A_{10} - A_{21})}{A_{10}(A_{20} + A_{21})}$$

All that is then required for population inversion is $A_{10} > A_{21}$. So the spontaneous emission into the 0 state must be more likely than into the 1 state. Then this system can achieve beam amplification