Phys 473 2015

Lecture 38

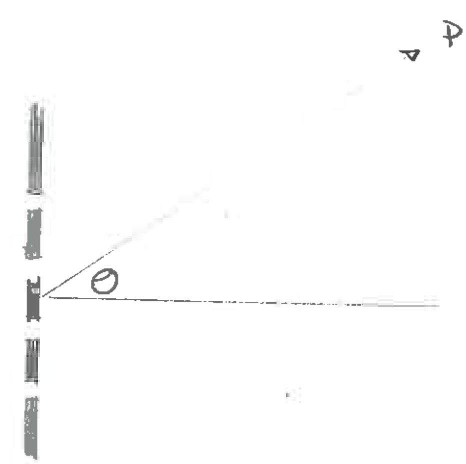
HW due Friday.

Multiple Slit Diffraction

We considered diffraction from an array of N equally spaced slits. With

$d =$ distance between centers of adjacent slits

$b =$ slit width



we found that at a distant point P

The intensity is

$$I(\theta) = I(0) \left[\frac{\sin\beta}{\beta}\right]^2 \left[\frac{\sin N\gamma}{N \sin\gamma}\right]^2$$

where

$$\beta = \frac{kb\sin\theta}{2}$$

and

$$\gamma = \frac{kd\sin\theta}{2}$$

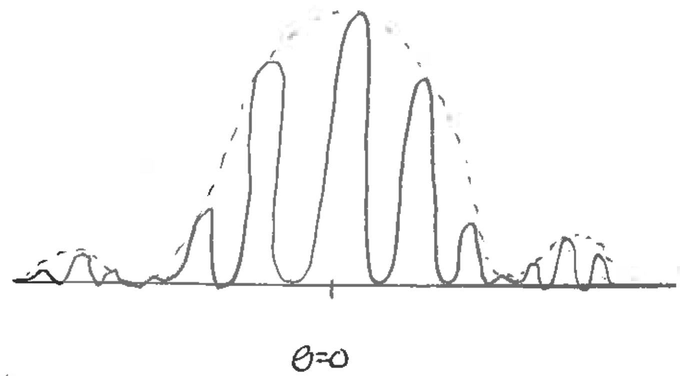
In general $b < d$ and thus the first term gives a slowly varying envelope to the intensity profile

$$\frac{\sin\beta}{\beta}.$$

The second term

$$\frac{\sin N\gamma}{N \sin\gamma}$$

gives a more rapidly varying term.



Certain minima will still occur at the minima of the envelope function. There will clearly be a maximum when $\theta = 0$.

We now consider locations of maxima of the rapidly varying term.

$$\left[\frac{\sin N\gamma}{N \sin\gamma}\right]^2$$

and ignore the envelope. We aim to find locations of such maxima and the values of intensity at these locations.

<u>Exercise:</u> a) Find a condition for the maximum of

$$\left[\frac{\sin N\gamma}{N \sin\gamma}\right]^2$$

b) Show that an extreme occurs when $\gamma = m\pi$ $\quad m =$ integer and evaluate the extreme. Use this to evaluate the maximal intensities I_0 in limit as slit width $\to 0$

<u>Answer:</u> a) Let

$$F(\gamma) = \left[\frac{\sin N\gamma}{N \sin\gamma}\right]^2$$

$$\frac{dF}{d\gamma} = 2\left[\frac{\sin(N\gamma)}{N \sin\gamma}\right] \frac{N^2 \cos N\gamma \sin\gamma - N \sin N\gamma \cos\gamma}{N^2 \sin^2\gamma}$$

We need

$$\frac{dF}{d\gamma} = 0$$

$$\Rightarrow \begin{cases} \sin(N\gamma) = 0 & \text{OR} \\ N\cos(N\gamma)\sin\gamma = \sin(N\gamma)\cos\gamma \end{cases}$$

b) Clearly we satisfy both of these when $\gamma = m\pi$. In this case

$$I = I(0)\left[\frac{\sin\beta}{\beta}\right]^2 \lim_{\gamma \to m\pi}\left[\frac{\sin N\gamma}{N\sin\gamma}\right]^2$$

We can ignore the envelope by letting $b \to 0 \Rightarrow \beta \to 0$. So in this limit.

$$I = I(0) \lim_{\gamma \to m\pi}\left[\frac{\sin N\gamma}{N\sin\gamma}\right]^2$$

Now the latter limit can be evaluated via

$$\lim_{\gamma \to m\pi} \frac{\sin N\gamma}{N\sin\gamma} = \lim_{\gamma \to m\pi} \frac{N\cos N\gamma}{N\cos\gamma} \to \frac{(-1)^{Nm}}{(-1)^m} = (-1)^N$$

Thus when $\gamma = m\pi$

$$I \to I(0)$$

$\boxtimes$

These maxima are called principle maxima as they all return the same maximum intensity. However, between these maxima we will find a succession of less pronounced maxima interspersed with minima.

<u>Exercise.</u> Determine values of γ that yield minima for

a) $N = 3$

b) $N = 4$

c) general N

<u>Answer:</u> Minima occur when

$$\sin N\gamma = 0$$

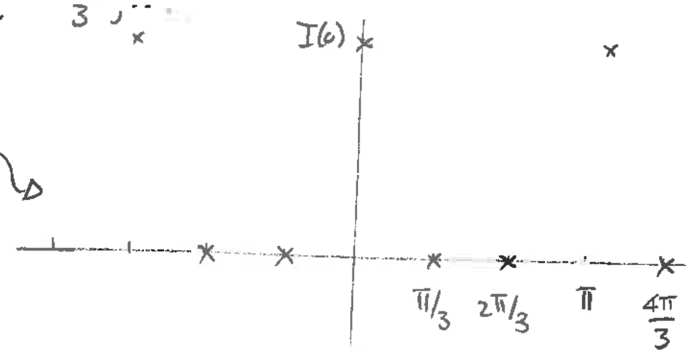
but

$$\sin \gamma \neq 0$$

The first occurs when $N\gamma = m\pi$ or $\gamma = \frac{m\pi}{N}$ but the latter requires that m is not a multiple of N

a) Here $\gamma = \pm \pi/3, \pm \frac{2\pi}{3}, \pm \frac{4\pi}{3}, \pm \frac{5\pi}{3}, \ldots$

We can plot maxima and minima at several locations:



b) $\gamma = \pm \pi/4, \pm \frac{2\pi}{4}, \pm \frac{3\pi}{4}, \pm \frac{5\pi}{4}, \ldots$

c) In general $\gamma = \pm \frac{\pi}{N}, \pm \frac{2\pi}{N}, \ldots \pm \frac{N-1}{N}\pi, \pm \frac{N+1}{N}\pi, \ldots$ ☒

Clearly these all satisfy the general conditions for minima. There must be some maxima located between these points.

<u>Exercise:</u> For the three slit case find locations other than $\gamma = m\pi$ for extrema

Find the intensity at those locations

<u>Answer:</u> $N\cos(N\gamma)\sin\gamma = \sin(N\gamma)\cos\gamma$

$$\Rightarrow 3\cos(3\gamma)\sin\gamma = \sin 3\gamma \cos\gamma$$

Clearly both sides equal zero when

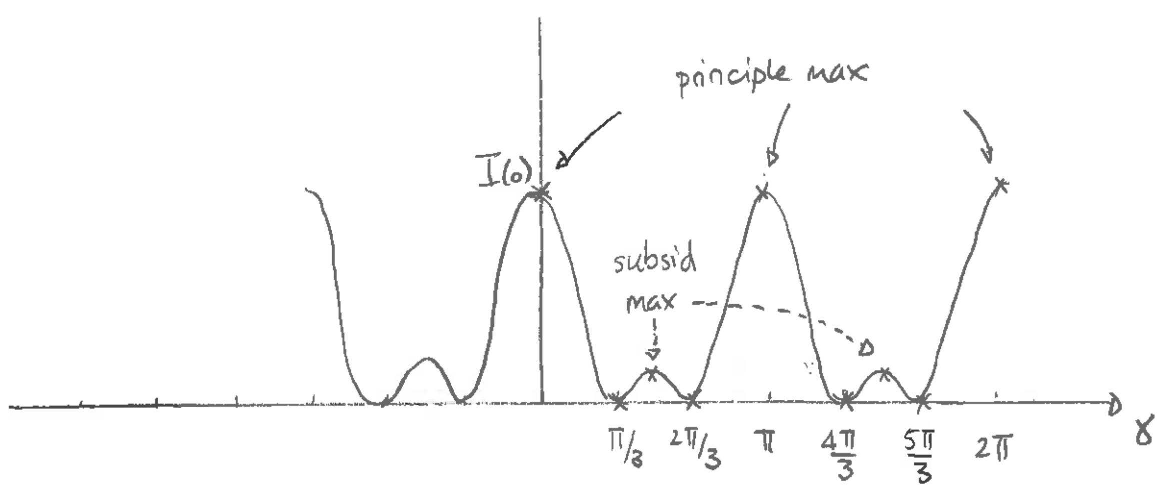
$$\gamma = \pm\pi/2,\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\ \ldots$$

At these locations (with $b \to 0$),

$$I = I(0)\left(\frac{\sin(N\gamma)}{N\sin\gamma}\right)^2 = I(0)\left[\frac{\sin(3\gamma)}{3\sin\gamma}\right]^2$$

$$\Rightarrow \quad I = I(0)\frac{1}{3^2} \Rightarrow \quad I = I(0)\frac{1}{9}$$

These maxima, which are much less intense are called subsidiary maxima.



In this case the condition for extrema gives:

$$3\cos 3\gamma \sin\gamma = \sin 3\gamma \cos\gamma$$

$$\cos 3\gamma = \cos 2\gamma \cos\gamma - \sin 2\gamma \sin\gamma$$

$$= (\cos^2\gamma - \sin^2\gamma)\cos\gamma - 2\sin\gamma\cos\gamma\sin\gamma$$

$$= \cos^3\gamma - 3\sin^2\gamma\cos\gamma$$

$$\sin 3\gamma = \sin 2\gamma\cos\gamma + \sin\gamma\cos 2\gamma$$

$$= 2\sin\gamma\cos^2\gamma + \sin\gamma(\cos^2\gamma - \sin^2\gamma)$$

$$= 3\sin\gamma\cos^2\gamma - \sin^3\gamma$$

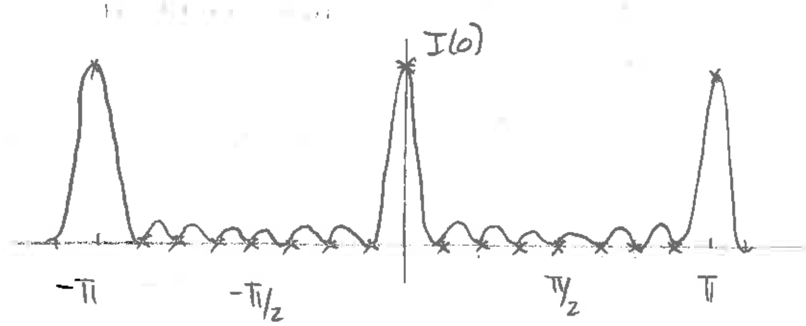
and the condition for an extremum is:

$$\cancel{3\sin\gamma\cos^3\gamma} - 9\sin^3\gamma\cos\gamma = \cancel{3\sin\gamma\cos^3\gamma} - \sin^3\gamma\cos\gamma$$

$$\Rightarrow \sin^3\gamma\cos\gamma = 0 \quad \Rightarrow \quad \cos\gamma = 0$$
$$\sin\gamma = 0$$

and we have thus found all possibilities for $N = 3$.

This analysis can be extended to more slits. The results regarding principle maxima and zeros apply just as well as before. The other maxima are not located as easily.



$N = 8$

Now when N is suitably large, for much of the central region between $\gamma = 0$, $\gamma = \pi$, $\sin\gamma \approx 1$ and maxima will be attained when $\sin N\gamma = 1$.

Clearly the maximum intensity will be about $\frac{1}{N^2} I(0)$ in this range.

We see that peaks become more pronounced and narrower.

To summarize:

For N slits principle maxima occur when

$$\gamma = m\pi \iff d \sin\theta = m\lambda$$

and the intensity at the principle maxima is

$$I = I(0)$$

There are minima at

$$\gamma = \pm\frac{\pi}{N}, \pm\frac{2\pi}{N}, \ldots \pm\frac{N-1}{N}\pi, \pm\frac{N+1}{N}\pi, \ldots$$ (excluding any integer multiples of N)

and the intensity at each minimum is

$$I = 0$$

There are subsidiary maxima at locations between the minima. There are N-2 subsidiary maxima between successive principle maxima and for most of these

$$I \to I(0)\frac{1}{N^2}$$

Diffraction gratings

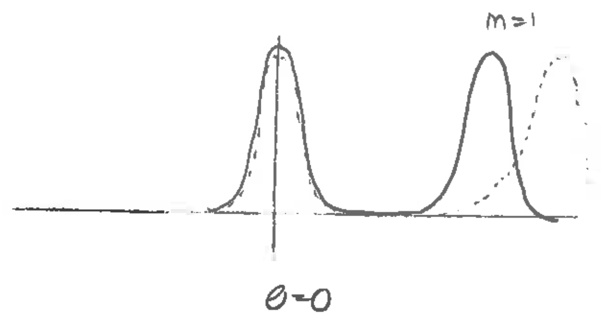
In a diffraction grating this is taken to the extreme as $N \to \infty$ and

As $N \to \infty$ the intensity everywhere approaches zero except at the principle maxima which are still located at

$$d \sin\theta = m\lambda$$

$m = \pm 1, \ldots$

Real gratings have a finite number of lines. This means that principle maxima will have non-zero width. If there are two distinct wavelengths of light entering the grating then it may not be possible to distinguish them at the principle maxima labeled m. To distinguish or resolve them, we require that the principle maximum of one must at least coincide will the nearest zero of the other. Now



$$\frac{d}{m} \sin\theta = \lambda$$

means that the angle at which the peak found for wavelength $\lambda + \Delta\lambda$ is $\theta + \Delta\theta$ where

$$\Delta\lambda = \frac{\partial\lambda}{\partial\theta}\Delta\theta \Rightarrow \Delta\lambda = \frac{d}{m}\cos\theta \; \Delta\theta$$

But $d/m = \frac{\lambda}{\sin\theta} \Rightarrow \Delta\lambda = \frac{\lambda}{\sin\theta}\cos\theta \; \Delta\theta$

$$\Rightarrow \frac{\Delta\lambda}{\lambda} = \frac{\cos\theta}{\sin\theta}\Delta\theta$$

Now we need $\Delta\theta$ to be the angular distance from a principle max to an adjacent minimum.

The principle max occurs when

$$\gamma = m\pi \quad \Rightarrow \quad d\sin\theta = m\lambda$$

and the adjacent minimum when

$$\gamma = m\pi + \frac{1}{N}\pi \quad \Rightarrow \quad \frac{\pi d}{\lambda}\sin(\theta + \Delta\theta) = m + \frac{1}{N}$$

$$\Rightarrow \quad d\sin(\theta + \Delta\theta) = \left(m + \frac{1}{N}\right)\lambda$$

But $\sin(\theta + \Delta\theta) \approx \sin\theta + \cos\theta \, \Delta\theta$

$$\Rightarrow \quad \underbrace{d\sin\theta + d\,\Delta\theta\cos\theta = m\lambda}_{\text{equal}} + \frac{\lambda}{N}$$

$$\Rightarrow \quad d\,\Delta\theta\cos\theta = \lambda/N$$

$$\Rightarrow \quad \Delta\theta\cos\theta = \frac{\lambda}{Nd}$$

Thus $\dfrac{\Delta\lambda}{\lambda} = \dfrac{\lambda}{Nd\sin\theta}$ and $d\sin\theta = m\lambda$ implies:

The minimum difference in wavelengths to resolve the m^{th} order principle maxima satisfies:

$$\boxed{\frac{\Delta\lambda}{\lambda} = \frac{1}{Nm}}$$