Diffraction from Multiple Slits

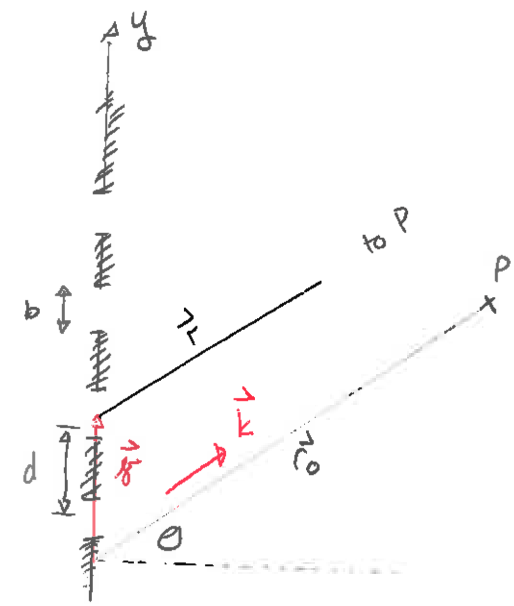
We can extend the single and double slit cases to the situation where there are multiple equally spaced slits, each with the same width.

Suppose that there are:

a) N slits

b) the centers of adjacent slits are separated by distance, d

c) each slit has width b



Again the Fraunhofer diffraction pattern is described via the electric field at point P and

$$E_p = C \int_{\text{aperture}} e^{ikr}\, dA$$

As before

$$kr = \vec{k}\cdot\vec{r}_0 - \vec{k}\cdot\vec{\rho}$$

and this gives:

$$E_p = C\, e^{i\vec{k}\cdot\vec{r}_0} \int_{\vec{\rho}\text{ spanning aperture}} e^{-i\vec{k}\cdot\vec{\rho}}\, dA.$$

Now using the illustrated geometry

$$\vec{k}\cdot\vec{r} = kr\cos(90^\circ-\theta) = kr\sin\theta$$

Placing the origin at the bottom of the lowest aperture gives the following range for $\vec{r} = x\hat{x} + y\hat{y}$

$$0 \leq x \leq L$$

$$0 \leq y \leq b$$

$$d \leq y \leq d+b$$

$$2d \leq y \leq 2d+b$$

$$(N-1)d \leq y \leq (N-1)d+b.$$

<u>Exercise</u>: Using this set up evaluate E_p and the intensity..

<u>Ans</u>: $$E_p = Ce^{i\vec{k}\cdot\vec{r}_0}L\left\{\int_0^b e^{-iky\sin\theta}dy + \int_d^{d+b} e^{-iky\sin\theta}dy + \dots + \int_{d(N-1)}^{d(N-1)+b} e^{-iky}dy\right\}$$

$$= \frac{Ce^{i\vec{k}\cdot\vec{r}_0}L}{-ik\sin\theta}\left\{ e^{-iky\sin\theta}\Big|_0^b + e^{-iky}\Big|_d^{d+b} + \dots + e^{-iky}\Big|_{d(N-1)}^{d(N-1)+b}\right\}$$

$$= \frac{Ce^{i\vec{k}\cdot\vec{r}_0}L}{-ik\sin\theta}\left(e^{-ikb\sin\theta}-1\right)\left\{1 + e^{-ikd\sin\theta} + \dots + e^{-ik(N-1)d\sin\theta}\right\}.$$

Then the geometric sum result is:

$$\sum_{k=0}^{n} ar^k = a\left(\frac{1-r^{k+1}}{1-r}\right)$$

So

$$E_p = \frac{C e^{i\vec{k}\cdot\vec{r}_0} L}{-ik\sin\theta}\left[e^{-ikb\sin\theta} - 1\right] \frac{1 - e^{-ikd\sin\theta\, N}}{1 - e^{-ikd\sin\theta}}$$

Now

$$e^{i\alpha} - 1 = e^{i\alpha/2}\left(e^{i\alpha/2} - e^{-i\alpha/2}\right)$$

$$= 2i\, e^{i\alpha/2} \sin \alpha/2$$

can be applied several times. Thus

$$E_p = \frac{C e^{i\vec{k}\cdot\vec{r}_0} L}{-\cancel{i}k\sin\theta} 2\cancel{i} e^{-i\frac{kb\sin\theta}{2}} \sin\left[\frac{-kb\sin\theta}{2}\right]$$

$$\times \frac{\cancel{2i} e^{-i\, kd\sin\theta\, N/2} \sin\left[\frac{-N\, kd\sin\theta}{2}\right]}{\cancel{2i}\, e^{-i\, kd\sin\theta} \sin\left[\frac{-kd\sin\theta}{2}\right]}$$

Let $\beta = \frac{kb\sin\theta}{2}$

$\gamma = \frac{kd\sin\theta}{2}$

Then $$E_p = -C e^{i\vec{k}\cdot\vec{r}_0} L\, e^{-ikb\sin\theta/2} e^{-ikd\sin\theta\,(N/2-1)}\left[\frac{\cancel{2}}{\cancel{2}\beta/b} \sin\beta\right] \frac{\sin N\gamma}{\sin\gamma}$$

Then $I = \frac{1}{2}\epsilon v\, E_p E_p^*$

$$= \frac{1}{2}\epsilon v\, |C|^2 L^2 b^2 \left[\frac{\sin\beta}{\beta}\right]^2 \left[\frac{\sin N\gamma}{\sin\gamma}\right]^2$$

Now as $\theta \to 0$ $\gamma \to 0$ $\Rightarrow$ $\sin N\gamma \to N\gamma$

$\sin \gamma \to \gamma$

$$\Rightarrow \quad \frac{\sin N\gamma}{\sin \gamma} \to N^2$$

Thus $I(0) = \frac{1}{2} \epsilon v |c|^2 L^2 b^2 N^2$ and this gives

For N equally spaced slits, each with width b and whose centers are separated by d, the intensity is

$$I(\theta) = I(0) \left[\frac{\sin\beta}{\beta}\right]^2 \left[\frac{\sin(N\gamma)}{N\sin\gamma}\right]^2$$

where $\beta = \dfrac{kb\sin\theta}{2}$

$\gamma = \dfrac{kd\sin\theta}{2}$

If $d \gg b$, then the pattern is described by an envelope $\left[\frac{\sin\beta}{\beta}\right]^2$ and an interference term $\left[\frac{\sin N\gamma}{N\sin\gamma}\right]^2$. We typically focus on the interference term. Within this the principle maxima occur when $\sin\gamma \to 0$. Thus we require

$$\gamma = m\pi \qquad m = \pm1, \pm2, \pm3, \ldots$$

$$\Rightarrow \quad kd\sin\theta = 2m\pi \quad \Rightarrow \quad \frac{2\pi}{\lambda} d\sin\theta = 2m\pi$$

$\Rightarrow$ $\boxed{\text{Principle maxima occur when } d\sin\theta = m\lambda \qquad m = \pm1, \pm2, \ldots}$

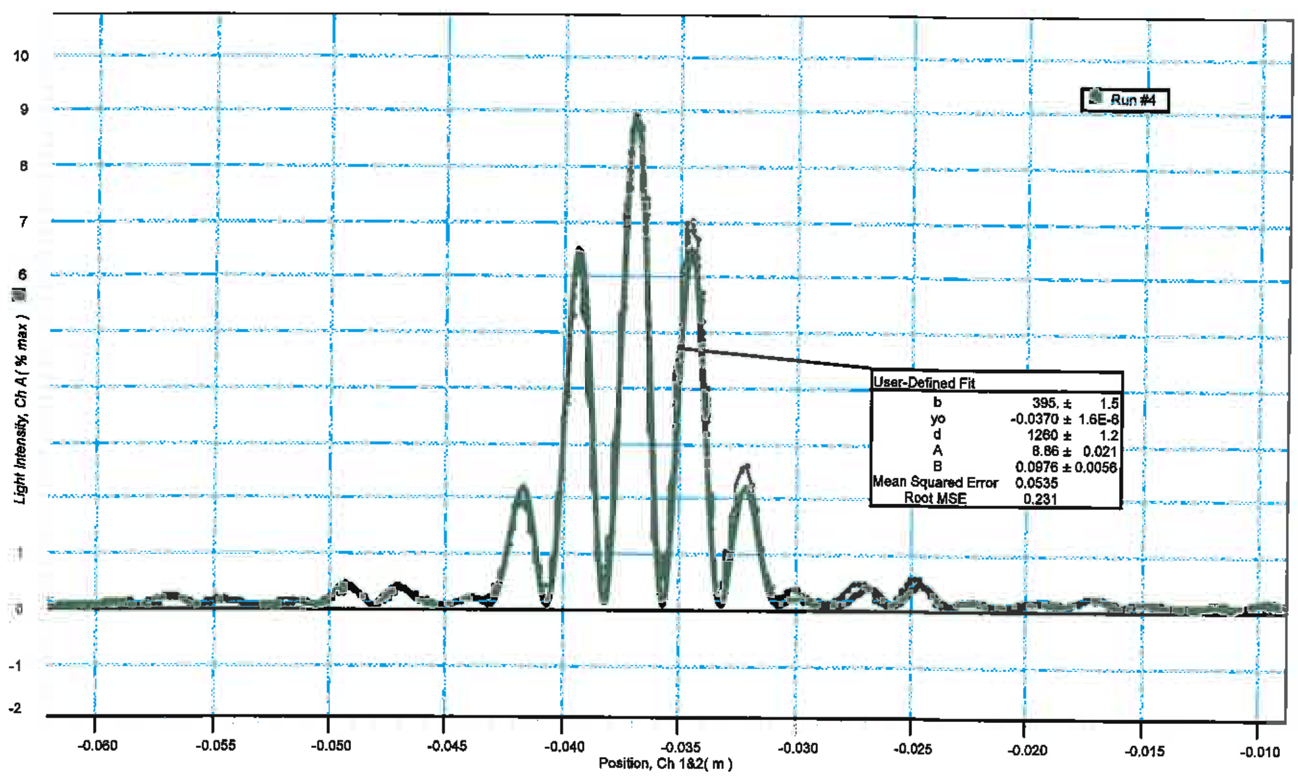
Lab Exercise: Using the given diffraction grating, determine the wavelength of the diode laser

Lab Exercise: a) Repeat the double slit experiment, measure L, b, d.

b) Fit curve

$$A\left[\frac{\sin B(x-x_0)}{B(x-x_0)}\right]^2\left[\frac{\sin(2(x-x_0)C)}{2\sin((x-x_0)C)}\right]^2+D$$

Graph 1


Run #4
User-Defined Fit
b 395. ± 1.5
yo -0.0370 ± 1.6E-6
d 1260 ± 1.2
A 8.86 ± 0.021
B 0.0976 ± 0.0056
Mean Squared Error 0.0535
Root MSE 0.231
Light Intensity, Ch A (% max)
10
9
8
7
6
1
0
-1
-2
-0.060
-0.055
-0.050
-0.045
-0.040
-0.035
-0.030
-0.025
-0.020
-0.015
-0.010
Position, Ch 1&2(m)