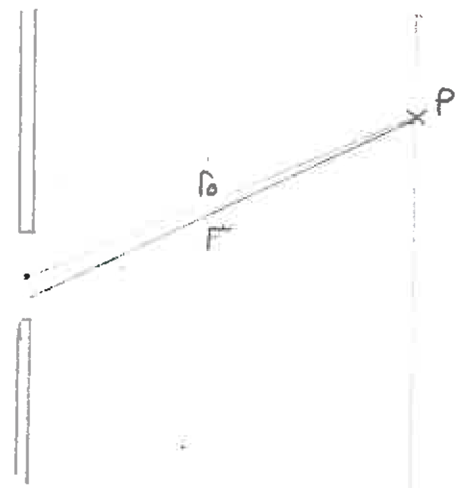
HW Friday!

Lab on Friday!

Fraunhofer Diffraction

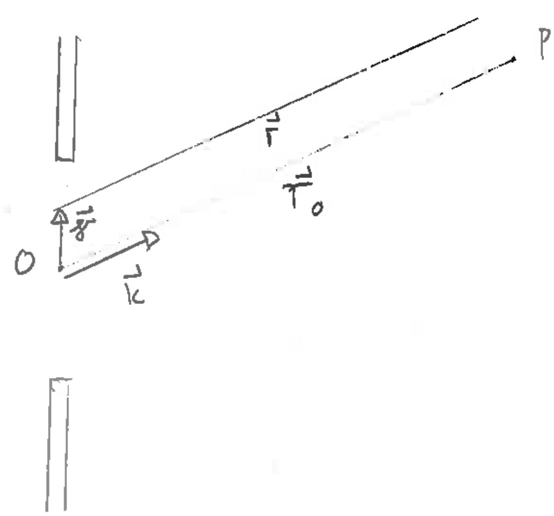
We have seen that at a field point P beyond any aperture, the electric field is

$$E_P = C \int_{\text{aperture}} e^{ikr}\, dA$$



where r varies as we sum over the aperture. We aim to rework this into a more algebraic rule suitable for any aperture.

In general we can describe the observation point via a vector from a reference point in the aperture. We consider waves that propagate along this direction. Then



$$kr = \vec{k}\cdot\vec{r}$$

But $\vec{r}_0 = \vec{r} + \vec{s}$ where $\vec{s}$ is as illustrated. So

$$kr = \vec{k}\cdot\vec{r}_0 - \vec{k}\cdot\vec{s}$$

Thus:

$$E_p = C \int_{\substack{\text{all vectors } \vec{\rho} \\ \text{across the aperture}}} e^{i(\vec{k}\cdot\vec{r}_0 - \vec{k}\cdot\vec{\rho})}\, dA$$

$$E_p = C e^{i\vec{k}\cdot\vec{r}_0} \int e^{-i\vec{k}\cdot\vec{\rho}}\, d\vec{A}$$

Since $\vec{k}$ and $\vec{r}_0$ are along the same direction $\vec{k}\cdot\vec{r}_0 = k r_0$ and this gives:

$$E_p = C e^{ikr_0} \int e^{-i\vec{k}\cdot\vec{\rho}}\, d\vec{A}$$

Also note that $\vec{k} = \vec{r}_0 \dfrac{k}{r_0}$

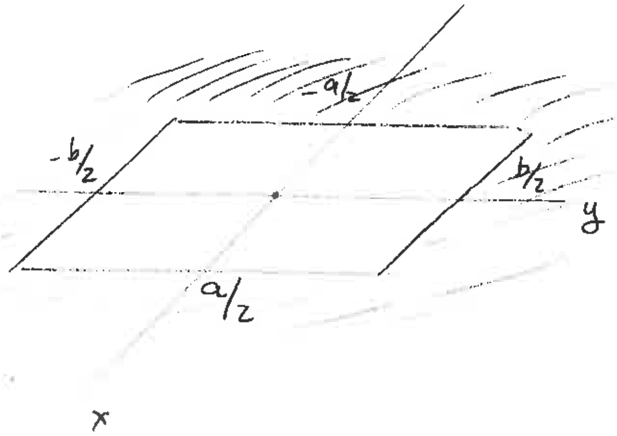
<u>Exercise:</u> Consider a rectangular aperture for which $-\frac{a}{2} \le x \le a/2$ and $-\frac{b}{2} \le y \le b/2$. Suppose that the reference point is at the center of the rectangle.

a) Describe the possible vectors $\vec{\rho}$ across the aperture

b) Given $\vec{r}_0$, this establishes

$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$

Use this to set up an integral for and evaluate E_p.



c) Let I_0 be the intensity above the midpoint. Determine $\vec{k}$ for this situation and use the result to determine an expression for intensity.

<u>Answer:</u> a) $\vec{r} = x\hat{x} + y\hat{y}$ $\qquad -\frac{a}{2} \le x \le a/2$

$-b/2 \le y \le b/2$

b) $\vec{k}\cdot\vec{r} = k_x x + k_y y$

$\Rightarrow$

$$E_p = Ce^{ikr_0} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \; e^{-i(k_x x + k_y y)} \, dx\,dy$$

$$= Ce^{ikr_0} \int_{-a/2}^{a/2} e^{-ik_x x} \, dx \int_{-b/2}^{b/2} e^{-ik_y y} \, dy$$

Then

$$\int_{-a/2}^{a/2} e^{ik_x x} = \frac{1}{ik_x} e^{-ik_x x}\Big|_{-a/2}^{a/2}$$

$$= \frac{1}{-ik_x}\left[e^{-ik_x a/2} - e^{ik_x a/2}\right]$$

$$= \frac{1}{\cancel{i}k_x} 2\cancel{i} \sin\left[\frac{k_x a}{2}\right]$$

$$= \frac{a/2 \; 2}{k_x a/2} \sin\left[\frac{k_x a}{2}\right] = a\left[\frac{\sin\alpha}{\alpha}\right]$$

where $\alpha = k_x a/2$. Similarly

$$\int_{-b/2}^{b/2} e^{ik_y y} \, dy = b\left[\frac{\sin\beta}{\beta}\right]$$

where $\beta = k_y b/2$

Thus

$$E_p = C e^{ikr_0}\, a\left[\frac{\sin\alpha}{\alpha}\right] b\left[\frac{\sin\beta}{\beta}\right].$$

c) The intensity is $\quad I = \frac{1}{2}\epsilon v\, E_p \cdot E_p^*$

$$= \frac{1}{2}\epsilon v\, |C|^2 a^2 b^2 \left[\frac{\sin\alpha}{\alpha}\right]^2 \left[\frac{\sin\beta}{\beta}\right]^2$$

where $\quad \alpha = \frac{k_x a}{2} \qquad \beta = \frac{k_y b}{2}$

Above the center $\vec{k} = k\hat{z} \Rightarrow k_x = k_y = 0 \Rightarrow \alpha = \beta = 0.$

Thus

$$I_0 = \frac{1}{2}\epsilon v |C|^2 a^2 b^2 \underbrace{\lim_{\alpha\to 0}\left[\frac{\sin\alpha}{\alpha}\right]^2 \lim_{\beta\to 0}\left[\frac{\sin\beta}{\beta}\right]}_{=1}$$

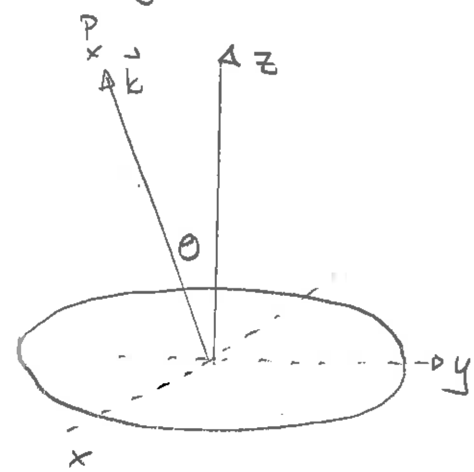
$$\Rightarrow \quad I_0 = \frac{1}{2}\epsilon v\, |C|^2 a^2 b^2$$

and this gives:

$$\boxed{I = I_0 \left[\frac{\sin\alpha}{\alpha}\right]^2 \left[\frac{\sin\beta}{\beta}\right]^2}$$

Diffraction above a circular aperture.

Consider a circular aperture of diameter D. The diffraction pattern will clearly be symmetrical about an axis through the center of the aperture. We aim to determine the intensity at an angle θ from the z-axis. We choose the co-ordinate system so that $\vec{k}$ lies in the x-z plane.



Thus

$$\vec{k} = k\cos\theta\,\hat{z} + k\sin\theta\,\hat{x}$$

<u>Exercise:</u> Set up an integral for the electric field at P.

<u>Answer:</u> The vector $\vec{s}$ lies in the xy plane. Thus

$$\vec{s} = s\cos\phi\,\hat{x} + s\sin\phi\,\hat{y}$$

where $0 \leq s \leq D/2$. So

$$\vec{k}\cdot\vec{s} = ks\cos\phi\sin\theta. \qquad \text{and} \quad dA = s\,ds\,d\phi$$

Then:

$$E_p = C\int_0^{D/2} s\,ds\int_0^{2\pi} d\phi\; e^{-iks\cos\phi\sin\theta}$$

To evaluate this integral we use:

$$\int_0^{2\pi} e^{-iz\cos\phi}\, d\phi = 2\pi J_0(z)$$

where $J_0(z)$ is a Bessel function of the first kind. This satisfies:

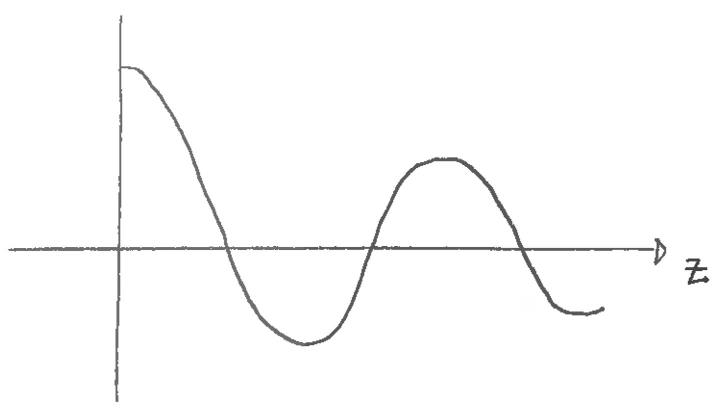
$$z^2 \frac{d^2 J_0}{dz^2} + z \frac{dJ_0}{dz} + z^2 J_0 = 0$$

and has series expansion

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+1)} \left(\frac{z}{2}\right)^{2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m} = 1 - \left(\frac{z}{2}\right)^2 + \frac{1}{4}\left(\frac{z}{2}\right)^4 + \ldots$$

This has a decaying oscillating character



Thus

$$E_p = C \int_0^{D/2} 2\pi J_0(ks\sin\theta)\, s\, ds$$

Another integral is

$$\int_0^r z J_0(z)\,dz = r J_1(r)$$

where $J_1(r)$ is another Bessel function:

$$J_1(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+1)!} \left(\frac{z}{2}\right)^{2m+1}.$$

and this also has an oscillatory character. letting $u = ks\sin\theta$ gives:

$$E_p = 2\pi c \int_0^{\frac{kD}{2}\sin\theta} J_0(u) \frac{u}{(k\sin\theta)^2}\,du$$

$$= \frac{\cancel{2}\pi c}{(k\sin\theta)^2} \frac{kD}{\cancel{2}} \cancel{\sin\theta}\, J_1\left(\frac{kD\sin\theta}{2}\right)$$

$$\Rightarrow E_p = \pi c D \frac{J_1\left(\frac{kD\sin\theta}{2}\right)}{\frac{kD\sin\theta}{2}} D/2$$

$$\Rightarrow E_p = \frac{\pi c}{2} D^2 \frac{J_1(\sigma)}{\sigma}$$

where $\sigma = \frac{kD\sin\theta}{2}$. It follows that

$$I(\theta) = c' \left[\frac{J_1(\sigma)}{\sigma}\right]^2$$

where c' is constant

Now $\frac{J_1(z)}{z} \to \frac{1}{2}$ as $z \to 0$. Thus

$$I(0) = \frac{c'}{4} \Rightarrow c' = 4\, I(0)$$

This gives:

For a circular aperature of diameter D,

$$I(\sigma) = I(0)\left[\frac{2J_1(\sigma)}{\sigma}\right]^2$$

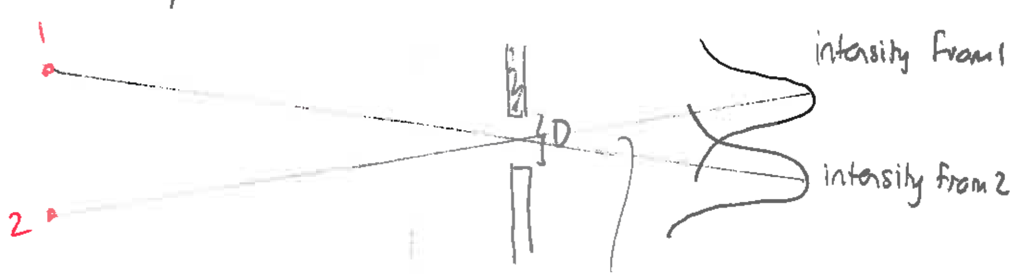
where $\sigma = \frac{kD\sin\theta}{2}$

The angular width of this pattern can be decided by finding the root of $J_1(\sigma)$. This occurs when $\sigma = 3.832$. Thus it occurs when

$$\sin\theta = \frac{2\times 3.832}{kD} = \frac{2\times 3.832}{2\pi}\,\frac{\lambda}{D} = 1.22\,\lambda/D$$

and this gives a spread, $\Delta\theta$.

This establishes the optical resolution of an optical system with a circular aperture.



must be larger than $\Delta\theta$