Lecture 34

Mon: Exam Ch 4,5

Review HW since last exam

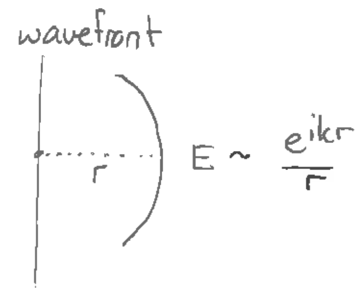
HW 11-21

Bring - another ½ letter sheet single side

Diffraction

Huygen's principle states that each point on a wavefront acts as a source of a wave that propagates spherically outward.

Thus if a plane wave is incident on an aperture in such a way that the wavefronts are parallel to the plane of the aperture, then at any location the field is

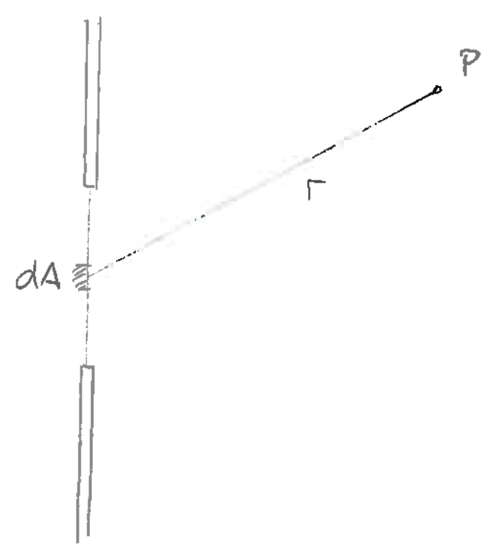


$$E_p = D \int \frac{e^{ikr}}{r} \, dA$$

where D is a constant,

dA is an area element,

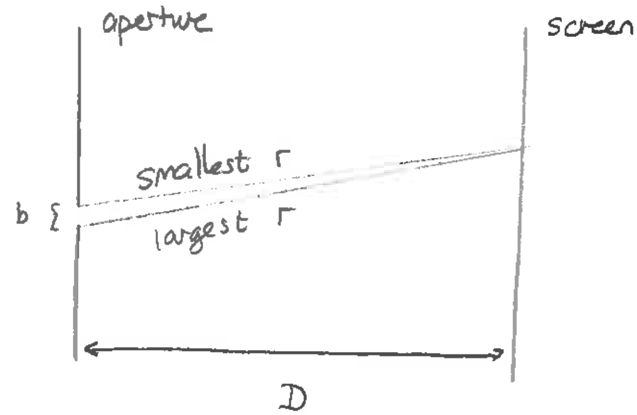
$k = \frac{2\pi}{\lambda}$



and the integral is over the aperture. As the dA element sweeps over the aperture, the distance r varies. This makes the integral difficult to evaluate in general.

Fraunhofer Diffraction

In Fraunhofer diffraction the interference pattern is observed on a screen that is very far from the aperture compared to the opening of the aperture. Intuitively this means that r does not vary much over the aperture opening.



Consider the two terms in the integral. First

$$f(r) = \frac{1}{r}$$

varies as

$$\Delta f = \frac{df}{dr}\Delta r = -\frac{1}{r^2}\Delta r$$

$$\Rightarrow \Delta f/f = -\frac{\Delta r}{r}$$

and if $\Delta r \ll r$, which is true as $D \gg b$ then $\Delta f/f \ll 1$ and variations in f are negligible. Second,

$$g(r) = e^{ikr}$$

varies as

$$\Delta g = \frac{dg}{dr}\Delta r \quad \Rightarrow \quad \Delta g = e^{ikr}\, ik\Delta r \Rightarrow \frac{\Delta g}{g} = ik\Delta r$$

$$\Rightarrow \frac{\Delta g}{g} = i\frac{2\pi}{\lambda}\Delta r$$

Even though Δr might be small compared to D it could be appreciable compared to λ. Thus $\Delta g/g$ could be significant. It follows that

> If $D \gg b$ then $1/r$ is approximately constant but e^{ikr} could vary significantly.

Thus we reach the key rule of Fraunhofer diffraction. Using the previous set up:

$$\boxed{E_p = C \int_{\substack{\text{aperture} \\ (r \text{ varies})}} e^{ikr} \, dA}$$

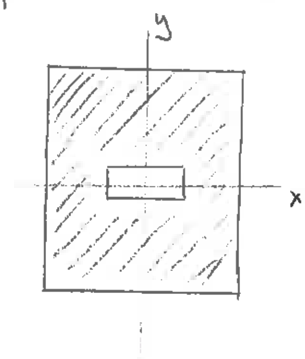
This is the Fraunhofer diffraction integral.

Single slit diffraction

The simplest form of aperture is a single rectangular opening, which we orient in the x-y plane. We assume that the aperture encompasses

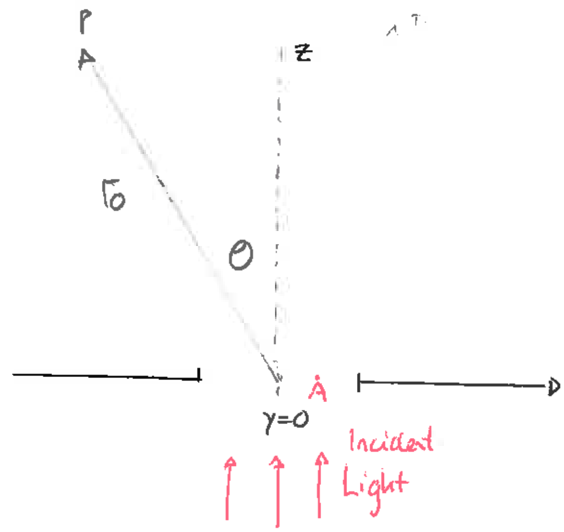
$$-\frac{a}{2} \leq x \leq a/2$$

$$-\frac{b}{2} \leq y \leq b/2$$



We initially focus on the y-z plane

Ignore the x dimension



We suppose that the observation point P is located at angle θ and distance r_0 from the midpoint of the aperture.

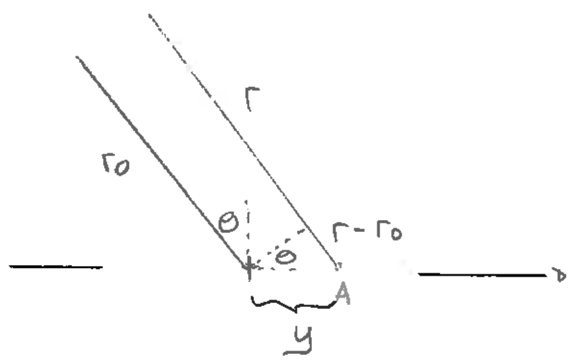
We start by considering the contribution of A to the field at P, extend this to all points and then integrate

Exercise: a) Express the distance, r, from point A to P in terms of r_0, θ and, y.

b) Use the previous result to set up an integral for the field at P

c) Evaluate the integral and use the result to obtain an expression for the intensity at point P

Answer a)



The rays are parallel and we see that

$$\frac{r - r_0}{y} = \sin\theta$$

$$\Rightarrow r = r_0 + y\sin\theta$$

b) $e^{ikr} \leadsto e^{ik(r_0 + y\sin\theta)}$ $\qquad dr = dy$

We get

$$E_p = C \int_{-b/2}^{b/2} e^{ikr_0} e^{iky\sin\theta}\, dy \int dx$$

and suppose that the x dimension does not contribute. Then let

$\int dx = L$

Thus

$$E_p = CLe^{ikr_0} \int_{-b/2}^{b/2} e^{iky\sin\theta}\, dy$$

c) $$E_p = CL\, e^{ikr_0} \frac{1}{ik\sin\theta} e^{ik\sin\theta} \Big|_{-b/2}^{b/2}$$

$$= CL\, e^{ikr_0} \frac{1}{ik\sin\theta} 2\sin\left[k\sin\theta\, b/2\right]$$

Then $$I = \epsilon v |E_p|^2$$

$$= 4\epsilon v C^2L^2 \frac{\sin^2\left[k\sin\theta b/2\right]}{(k\sin\theta)^2}$$

We now rewrite this result in terms of a new parameter

$$\beta := \frac{kb\sin\theta}{2}$$

Then

$$I = 4\epsilon v\, C^2L^2 \frac{\sin^2\beta}{(2\beta/b)^2}$$

$$= \epsilon v\, C^2L^2\, b^2 \left[\frac{\sin\beta}{\beta}\right]^2$$

Now at the point perpendicular to the center of the aperture, $\theta = 0$ and this gives $\beta \to 0$. We know $\frac{\sin\beta}{\beta} \to 1$ as $\beta \to 0$. Thus with

I_0 = intensity when $\theta = 0$

We get the main result:

For a single slit of width β, the intensity for $r \gg b$ is

$$I = I_0 \left[\frac{\sin\beta}{\beta}\right]^2$$

where

I_0 = intensity at $\theta = 0$

$$\beta = \frac{kb\sin\theta}{2}$$

Exercise: Determine angles at which the intensity is

a) a minimum (in terms of λ, ...

b) a maximum

Answer a) Except for $\beta = 0$ we need

$$\sin\beta = 0$$

$$\Rightarrow \beta = n\pi \qquad n = \pm 1, \pm 2, \ldots$$

$$\Rightarrow \frac{kb\sin\theta}{2} = n\pi \Rightarrow \frac{\cancel{2}\cancel{\pi}}{\lambda}\frac{b\sin\theta}{\cancel{2}} = n\cancel{\pi}$$

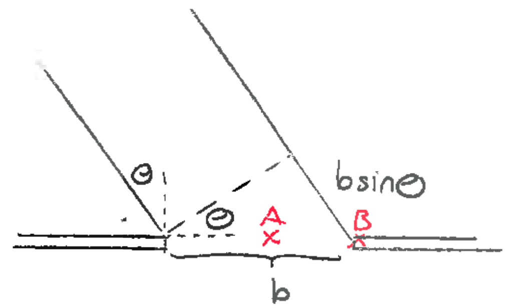
$$\Rightarrow b\sin\theta = n\lambda$$

b) It might appear that $\sin\beta = \pm 1$ yields a maximum but this is only approximate. We need

$$\frac{\partial}{\partial\beta}\frac{\sin\beta}{\beta} = 0 \Rightarrow \frac{\cos\beta}{\beta} - \frac{\sin\beta}{\beta^2} = 0$$

$$\Rightarrow \beta\cos\beta = \sin\beta \Rightarrow \beta = \tan\beta$$

There is an intuitive explanation for the locations of the minima.
At this angle the path difference from the extremes is $b\sin\theta$.
If this is a single wavelength then the waves emanating from the extreme points are in phase. But those emanating from the midpoint are out of phase with B. These cancel.

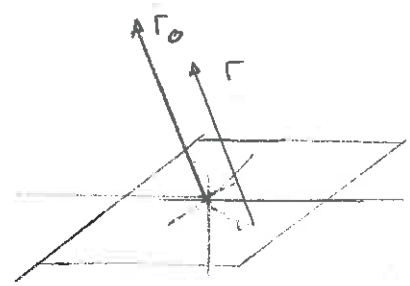


Such pairwise cancellation occurs for all pairs separated by $b/2$. Thus all waves cancel.

Rectangular aperature.

For a rectangular aperture, the two variables can be separated. The resulting integral gives:

$$I = I_0 \left(\frac{\sin\alpha}{\alpha}\right)^2 \left(\frac{\sin\beta}{\beta}\right)^2$$



where now

$$\alpha = \frac{ka\sin\phi}{z}$$

and ϕ is the angle in the x-z plane