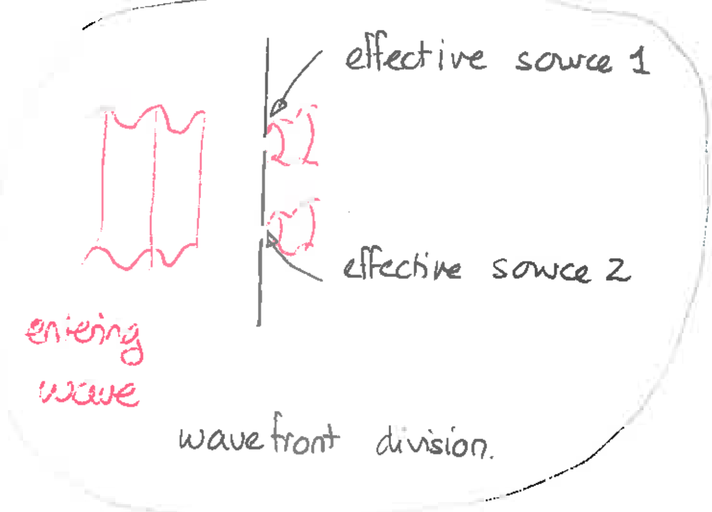
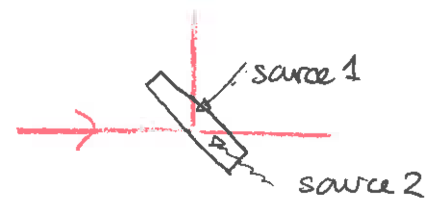
Exam II Monday 9 Nov Ch 4,5

Diffraction

So far we have considered interference resulting from two or more distinct sources. Such sources could be produced by wavefront division as in multiple slit experiments. An alternative description involves amplitude division which is often attained via a beamsplitter





Both of these are analyzed using the mathematics of adding two waves, typically:

source 1 $\rightsquigarrow$ $E_1(x,t)$

source 2 $\rightsquigarrow$ $E_2(x,t)$

$\Longrightarrow$ wave at detector

$E(t) = E_1(t) + E_2(t)$

$\hookrightarrow$ Intensity at detector

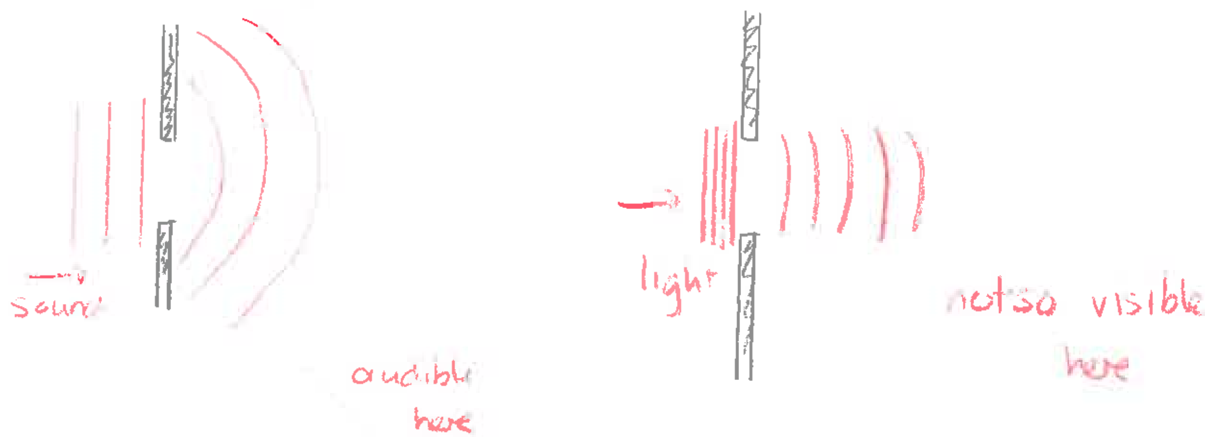
$I = \epsilon v E E^*$

Now consider light passing through a single aperture.

Demo: PASCO laser / aperture. – single slit.

This is an example of diffraction, an interference phenomenon, in which waves from a single aperature can produce interference patterns. Such diffraction occurs for all types of waves and can be used to explain:

1) propagation of waves around barriers.



2) limitations of transmission of and detection of waves

3) limitations of optical instruments

Broadly the study of diffraction involves:

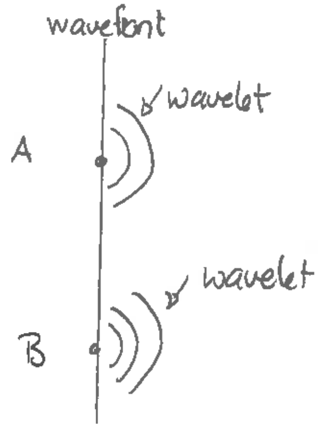
1) a general principle, Huygen's principle

2) a specialization of the case where the detector / screen is "far" from the aperture – this is Fraunhofer diffraction.

3) a specialization of the case where the detector/screen is "close" to the aperture – this is Fresnel diffraction.

Huygen's principle

Huygen's principle states that every point on a wavefront acts as a source for a secondary wavelet that propagates outward from that point. In the illustration, point A acts as a source of the indicated wavelet. The overall wave disturbance is attained as a superposition over all such wavelets from all points on the wavefront. These radiate outward with diminishing amplitude and we can sketch the wavefront produced by all points:



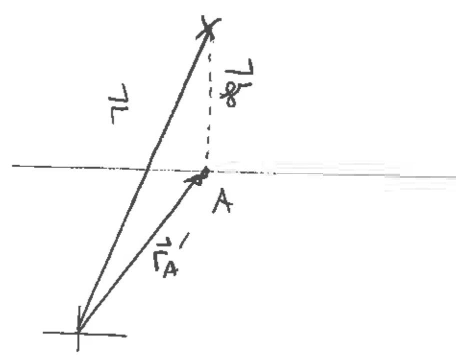
To analyze this consider the fields produced at location $\vec{r}$. The conceptually

$$E(\vec{r}) = \sum E_A(\vec{r})$$

where

$$E_A(\vec{r})$$

is the field produced by source A at the field point. This is

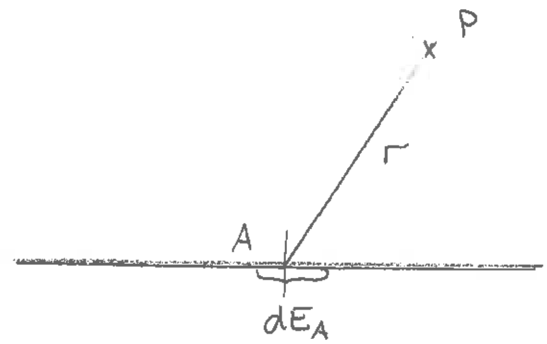


$$E_A(\vec{r}) = \frac{E_{oA}\, e^{i(k\varrho - \omega t)}}{\varrho}$$

where $\varrho = |\vec{r} - \vec{r}_A{}'|$ and E_{oA} is the field at point A.

Huygen's principle : plane wave front

Suppose that we consider a plane wavefront with all points in phase and with field amplitude E_0. Using the indicated co-ordinates the field at point P is schematically



$$E_p = \int_A \frac{1}{r} e^{i(kr-\omega t)} dE_A$$

where :

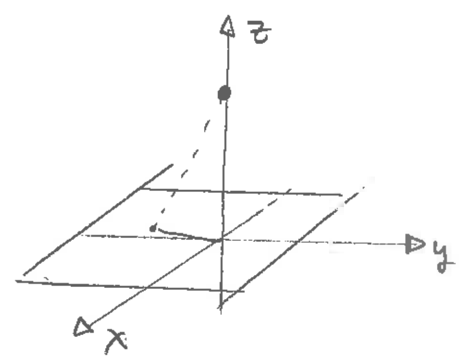
1) r varies as A sweeps over the surface
2) dE_A is the field from portion A

Then if the field is uniform dE_A = constant $\times E_0 \, dA$. This gives

$$E_p = \text{Const} \times E_0 \int_{\text{all plane A}} \frac{1}{r} e^{ikr} dA \; e^{-i\omega t}$$

Let the constant be D. Thus:

$$E_p = D \int_{\text{wavefront}} \frac{1}{r} e^{ikr} dA \; e^{-i\omega t}$$



For example, for a field produced above a wavefront in the x-y plane:

$$E_p = D \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{ik(x^2+y^2+z^2)^{1/2}}}{(x^2+y^2+z^2)^{1/2}} dx\,dy \; e^{-i\omega t}$$

We can consider the field at $t=0$. Taking the real part of E gives the real field and in this situation the field at $z=0$ attains a peak. We aim to find other locations of crests.

Then, with $t=0$

$$E_p = D \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \frac{e^{ik(x^2+y^2+z^2)^{1/2}}}{(x^2+y^2+z^2)^{1/2}}$$

We transform to radial co-ordinates,

$$x = r\cos\phi \qquad\qquad 0 \le r \le \infty$$
$$y = r\sin\phi \qquad\qquad 0 \le \phi \le 2\pi$$

Then

$$dx\,dy = r\,dr\,d\phi$$

$$x^2+y^2+z^2 = r^2+z^2$$

So

$$E_p = D \int_0^{\infty} r\,dr \int_0^{2\pi} d\phi \; \frac{e^{ik(r^2+z^2)^{1/2}}}{(r^2+z^2)^{1/2}}$$

$$= 2\pi D \int_0^{\infty} \frac{e^{ik(r^2+z^2)^{1/2}}}{(r^2+z^2)^{1/2}} \, r\,dr$$

Exercise: a) Rewrite the integral in terms of $u = \sqrt{r^2+z^2}$ (for $z \geq 0$)

b) Show that the result is periodic in z and determine the period

c) Determine values of z that give maxima + minima. (for a real field)

Answer: a) Here $0 \leq r \leq \infty \Rightarrow z \leq u \leq \infty$

$$du = \frac{r}{\sqrt{r^2+z^2}}\, dr \Rightarrow u\,du = r\,dr$$

The integrand is:

$$\frac{e^{ik(r^2+z^2)^{1/2}}}{(r^2+z^2)^{1/2}}\, r\,dr = \frac{e^{iku}}{u}\, u\,du$$

$$= e^{iku}$$

$$\Rightarrow E_p(z) = 2\pi D \int_z^\infty e^{iku}\, du = \frac{2\pi D}{ik}\, e^{iku}\Big|_z^\infty$$

This cannot be evaluated as the upper limit

$$\lim_{u \to \infty} 2\pi D\, e^{iku}$$

does not exist.

b) $$E_p(z+z') = \frac{2\pi D}{ik}\, e^{iku}\Big|_{z+z'}^\infty = \underbrace{\frac{2\pi D}{ik}\, e^{iku}\Big|_z^\infty}_{E_p(z)} + \frac{2\pi D}{ik}\, e^{iku}\Big|_{z'+z}^{z}$$

$$\Rightarrow E_p(z+z') = E_p(z) + \frac{2\pi D}{ik}\left[e^{ikz} - e^{ik(z'+z)}\right]$$

Then $\quad E_p(z+z') = E_p(z) \iff e^{ikz} = e^{ik(z'+z)}$

$$\iff e^{ikz'} = 1$$

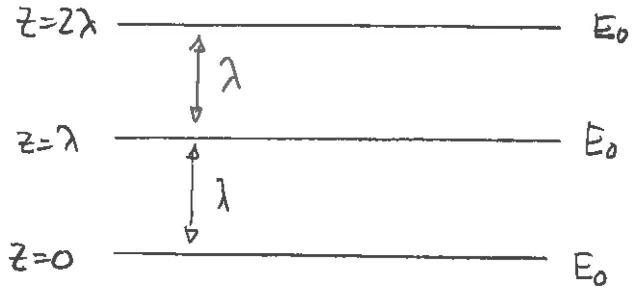
$$\iff kz' = 2n\pi$$

$$\iff \frac{\cancel{2}\cancel{\pi}}{\lambda} z' = \cancel{2} n \cancel{\pi}$$

$$\iff z' = n\lambda.$$

Thus the field repeats on planes parallel to the x-y plane. These planes are separated by multiples of the wavelength.

c) Knowing that the field attains a maximum when $z=0$ and that it is periodic according to the results of b)



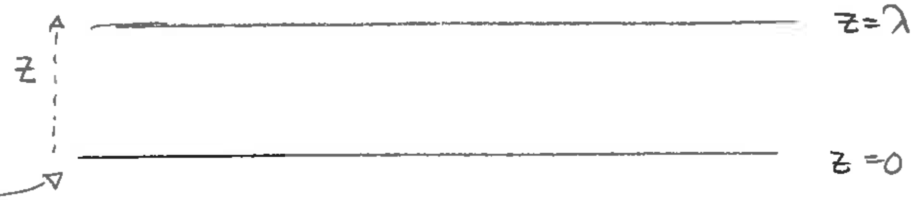
So maxima occur when

$z = n\lambda \qquad n = 0,1,2,3\ldots$

Minima occur at $z = \lambda/2, \frac{3\lambda}{2}, \ldots$

This also allows for wavelets propagating in reverse. The fact that this does not occur is explained with a correction in Bennett. A more thorough explanation is given in Kenyon

Thus we get



wavelets
emitted in phase
at all points on this surface

$\Rightarrow$ wavelets add to give field that is constant along planes parallel to x y plane

$\Rightarrow$ wavelets add so that electric field repeats on planes separated by λ

It follows that Huygen's principle is consistent with the way that plane waves propagate.