Phys 473 2015

Lecture 32

HW due Weds

Coherence of field.

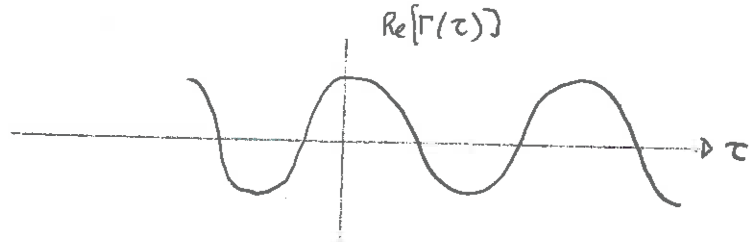
We saw that if the electric field is represented using complex exponentials then a measure of the correlations that exist between the field at various times is the coherence function

$$\Gamma(\tau) = \langle E(t)\, E^*(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} E(t')\, E^*(t'+\tau)\, dt'$$

Example: For a pure monochromatic field $E = E_0 e^{i\omega t}$,

$$\Gamma(\tau) = |E_0|^2 \cdot e^{i\omega\tau}$$

and we can plot $Re[\Gamma(\tau)] = |E_0|^2 \cos(\omega\tau)$



Exercise: Suppose that the field has a randomly varying phase, ie $E = E_0 e^{i[\omega t + \phi(t)]}$. Determine a general expression for $\Gamma(\tau)$. Then consider the case where $\phi(t)$ varies randomly many times in the interval τ. Determine $\Gamma(\tau)$ in this case.

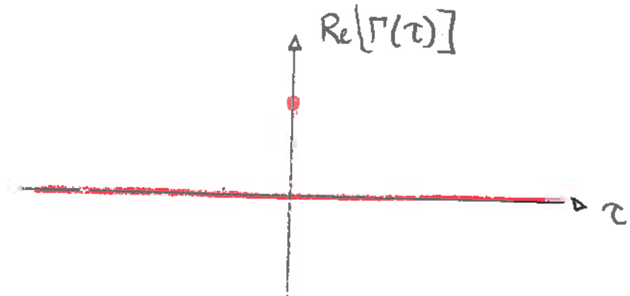
Answer:

$$\Gamma(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} |E_0|^2 e^{i[\omega t' + \phi(t')]} e^{-i(\omega t' + \omega\tau + \phi(t'+\tau))} dt'$$

$$= |E_0|^2 e^{i\omega\tau} \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} e^{i[\phi(t') - \phi(t'+\tau)]} dt'$$

The integrand contains $\cos[\phi(t') - \phi(t'+\tau)]$ and $i\sin[\dots]$. If the phase fluctuates rapidly during interval τ then both of these will average to zero. Thus $\Gamma(\tau) \to 0$

For this case



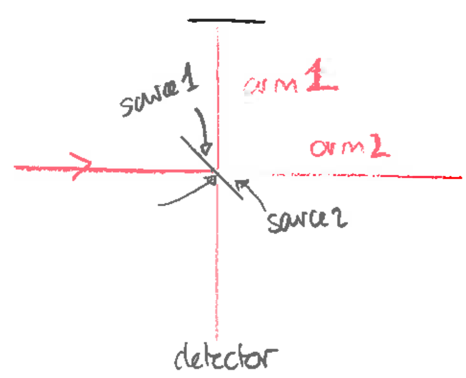
Here the coherence functions have the form

$$\Gamma(\tau) = |E_0|^2 e^{i\omega\tau} \times \text{function of } (t,\tau).$$

which indicates a sinusoidally oscillating function with magnitude $|E_0|^2$ multiplied by a function of t, τ. For monochromatic fields the oscillations are maximized. The extent of the oscillations will dictate the coherence.

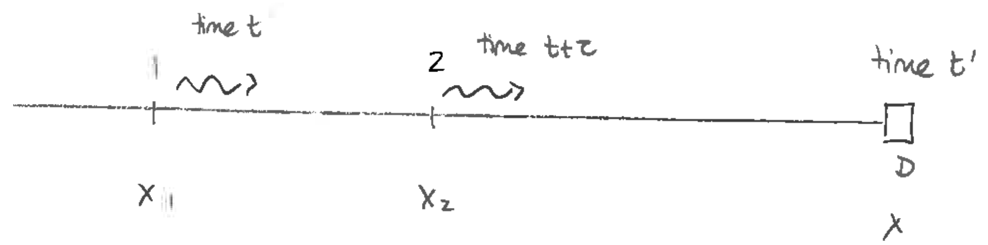
Coherence + interference

Consider a Michelson Interferometer with unequal arm lengths. Although we think of the input laser or light source as the source, the actual source for interferometry is the point A at the beam splitter:



Ignoring the reflection phase shifts. This acts as two sources one for arm 1 and the other for arm 2.

We want to compare the field produced by source 1 at time t to that produced by source 2 at time $t+\tau$. If $\tau>0$ then the field produced by 2 occurs at a later time and this must be closer to the detector. So we can envisage the sources as



Let the field produced by source 1 at the source be $E_{source\,1}(t)$ with a similar construction for the field produced by source 2. Then the fields at the detector at time t' are:

source 1 ⇝ $E_1(t') = E_{source\,1}\left(t' - \frac{x-x_1}{v}\right)$

$$E_2(t') = E_{source\,2}\left(t' - \frac{x-x_2}{v}\right)$$

with superposition

$$E(t') = E_1(t') + E_2(t')$$

let $t = t' - \frac{x-x_1}{v}$, which is the time when the field produced by source 1 was produced Thus $E_1(t') = E_{source\,1}(t)$. Now

$$t' - \frac{x-x_2}{v} = t' - \left[\frac{x-x_1}{v} + \frac{x_1-x_2}{v}\right]$$

But $\frac{x_2-x_1}{v} = \tau \Rightarrow t' - \frac{x-x_2}{v} = t' - \frac{x-x_1}{v} + \tau$ gives

$$E_2(t') = E_{source\,2}\left(t' - \frac{x-x_1}{v} + \tau\right) \Rightarrow E_2(t') = E_{source\,2}(t+\tau)$$

$$E_1(t') = E_{source\,1}(t)$$

Then the intensity at the detector is:

$$I = \epsilon v \langle E(t') E^*(t') \rangle$$

<u>Exercise</u>: Rewrite this in terms of the fields at the sources.

<u>Ans</u>: $$I = \epsilon v \langle [E_1(t') + E_2(t')][E_1^*(t') + E_2^*(t')] \rangle$$

$$= \epsilon v \langle E_1(t') E_1^*(t') \rangle$$
$$+ \epsilon v \langle E_2(t') E_2^*(t') \rangle$$
$$+ \epsilon v \langle E_1(t') E_2^*(t') \rangle$$
$$+ \epsilon v \langle E_1^*(t') E_2(t') \rangle$$

$$= \epsilon v \langle E_{source\,1}(t) E_{source\,2}^*(t) \rangle$$
$$+ \epsilon v \langle E_{source\,2}(t+\tau) E_{source\,2}^*(t+\tau) \rangle$$
$$+ \epsilon v \langle E_{source\,1}(t) E_{source\,2}^*(t+\tau) \rangle$$
$$+ \epsilon v \langle E_{source\,1}^*(t) E_{source\,2}(t+\tau) \rangle$$

Now the intensity of source 1 is

$$I_1 = \epsilon v \langle E_{source\,1}(t) E_{source\,1}^*(t) \rangle$$

with a similar rule for source 2.

Also $$E_{source\,1}^*(t) E_{source\,2}(t+\tau) = [E_{source\,1}(t) E_{source\,2}^*(t+\tau)]^*$$

These result

The intensity at the detector is.

$$I = I_1 + I_2 + 2\epsilon V \, Re[\Gamma_{12}(\tau)]$$

where I_1 is the intensity of source 1

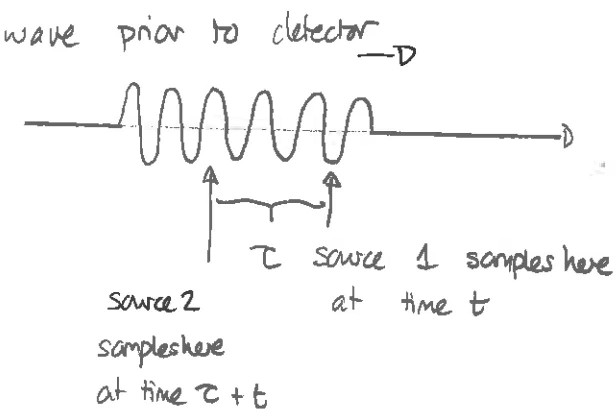
I_2 is the intensity of source 2

and

$$\Gamma_{12}(\tau) = \langle E_{source\,1}(t) \, E^*_{source\,2}(t+\tau) \rangle$$

is the mutual coherence function.

This captures the coherence information of the light beam prior to the beam splitter. Suppose that the wave prior to the beam splitter has form. Then



source 1 samples at one instant t.

Source 2 effectively samples at a later instant $t+\tau$.

We now rework the intensity of the superposition. We define the normalized degree of partial coherence as:

$$\gamma_{12}(\tau) = \epsilon V \frac{\Gamma_{12}(\tau)}{\sqrt{I_1 I_2}}.$$

Then:

$$\boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2}\, Re[\gamma_{12}]}$$

Now

$$\gamma_{12}(\tau) = \frac{\langle E_{source\,1}(t)\, E_{source\,2}^*(t+\tau)\rangle}{\sqrt{\langle E_1(t) E_1^*(t)\rangle \langle E_2(t) E_2^*(t)\rangle}}$$

and a general mathematical result gives:

1) $0 \leqslant |\gamma_{12}| \leqslant 1$

2) $\gamma_{12}(0) = 1$

<u>Exercise</u>: Determine $\gamma_{12}(\tau)$ for a monochromatic wave where

$$E_{source\,1} = E_{01} e^{i\omega t} \quad etc, \ldots$$

(E_{01} → real)

<u>Answer</u>:

$$E_{source\,1}(t)\, E_{source\,2}^*(t+\tau) = e^{-i\omega\tau} E_{01} E_{02}^*$$

$$E_{source\,1}(t)\, E_{source\,1}^*(t) = |E_{01}|^2$$

$$E_{source\,2}(t)\, E_{source\,2}^*(t) = |E_{02}|^2$$

$$\Rightarrow \gamma_{12}(\tau) = \frac{E_{01}}{|E_{01}|}\frac{E_{02}^*}{|E_{02}|} e^{-i\omega\tau} = e^{-i\omega\tau}$$

We see that $|\gamma_{12}(\tau)| = 1$ and that it oscillates with τ.

Exercise Show that the visibility for a monochromatic wave,

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

is

$$V = |\gamma_{12}|$$

whenever $I_1 = I_2$

Answer: $\gamma_{12} = |\gamma_{12}| e^{i\alpha(t)}$ where $\alpha(t) = \omega\tau$ for monochromatic waves. Thus

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}\,|\gamma_{12}|$$

$$I_{min} = I_1 + I - 2\sqrt{I_1 I_2}\,|\gamma_{12}|$$

$$\Rightarrow \quad V = \frac{4\sqrt{I_1 I_2}\,|\gamma_{12}|}{(I_1 + I_2)\,2}$$

For monochromatic waves $V = |\gamma_{12}|$

Coherence time.

The coherence time is defined as the time τ_c such that the visibility is half. The coherence length L_c is the distance traveled in this time $L_c = c\tau_c$. In practice these are related to wavelength spreads and frequency spreads. So

$$\Delta k \Delta x = 1$$

$$\Rightarrow \quad \boxed{\Delta k L_c = 1} \quad \text{and} \quad \boxed{\Delta\omega \tau_c = 1}$$