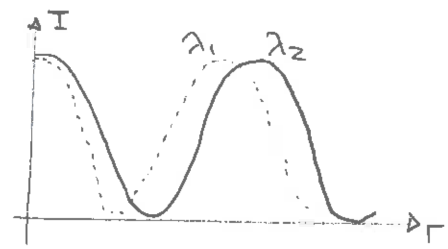
Phys 473 2015

Lecture 30

Multiple Beam Interference

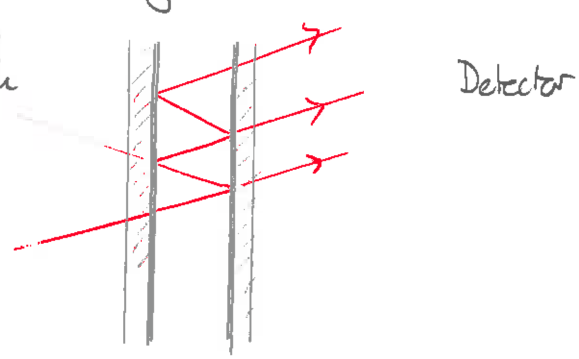
Certain interferometers use repeated interference to enhance the resolution of interference fringes. Recall that a Michelson interferometer produces an interference pattern consisting of circular fringes. Depending on the relative path length the intensity may be as illustrated – there will be broad areas which are generally bright. If two closely spaced wavelengths are present then it may be impossible to distinguish the peaks of these.



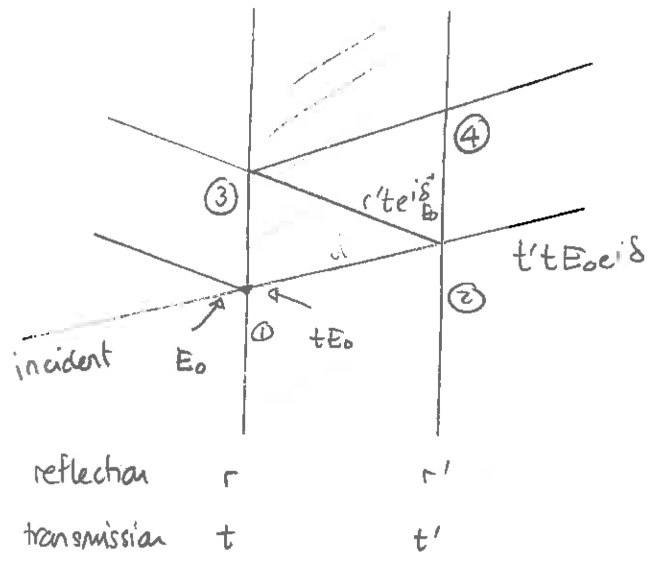
However, multiple interference sharpen the lines. One example of this is a diffraction grating (as compared to a double slit)

The diffraction grating produces much narrower peaks, allowing for resolution of peaks that were indistinguishable with a double slit

One interferometer version of this is the Fabry-Perot interferometer. This consists of two closely spaced partly transmitting elements. Repeated reflection + transmission produces many interfering beams.



A simplified model for this consists of two parallel surfaces. An incident beam is partly reflected + partly transmitted



* At point ①
  - reflected field is rE_0
  - transmitted field is tE_0

* At point ②
  - the incident field (from the left) is

$$tE_0 e^{i\tilde{\delta}}$$

where $\tilde{\delta} = \frac{d}{\lambda_n} 2\pi$ where λ_n is the wavelength in the medium between surfaces and d is the distance traveled. Then $\lambda_n = \frac{\lambda}{n}$ where λ is the wavelength in a vacuum, so

$$\tilde{\delta} = \frac{nd}{\lambda} 2\pi = nkd$$

  - the transmitted field is

$$t' t E_0 e^{i\tilde{\delta}}$$

  - the reflected field is $r' t E_0 e^{i\tilde{\delta}}$

We can continue thus:

<u>Exercise</u>: a) Determine the reflected + transmitted fields at ③

b) " " " " " ④

c) Use this to determine expressions for the first four beams that emerge at the right

Answers: a) the incident field at 3 is

$$r' t E_o e^{2i\tilde{\delta}}$$

So the transmitted field to the left is

$$t r' t E_o e^{2i\tilde{\delta}}$$

The reflected field is

$$r r' t E_o e^{2i\tilde{\delta}}$$

b) the incident field at 4 is:

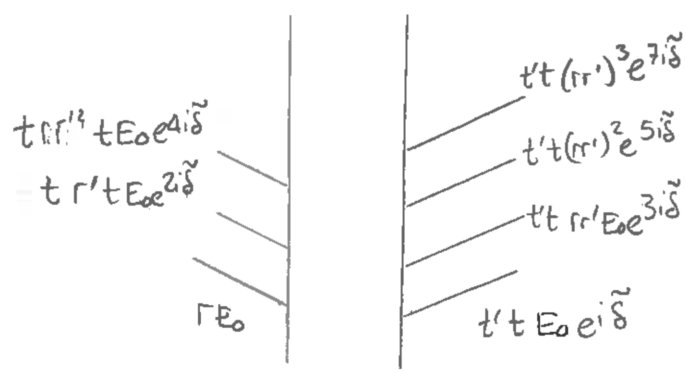
$$r r' t E_o e^{3i\tilde{\delta}}$$

the rightward transmitted field is $\quad t' r r' t e^{3i\tilde{\delta}}$

the leftward reflected field is $\quad r' r r' t e^{3i\tilde{\delta}} = r t r'^2 e^{3i\tilde{\delta}}$

c) each successive beam emerging on the right is attain from the previous reflected beam by

- an additional factor of r from the left side
- " " " of r' " " right side
- an " phase shift of $e^{2i\tilde{\delta}}$



Thus the field that emerges is:

$$E_t = t't E_0 e^{i\tilde{\delta}} + t't(rr') E_0 e^{3i\tilde{\delta}} + tt'(rr')^2 E_0 e^{5i\tilde{\delta}} + \dots$$

$$= t't E_0 e^{i\tilde{\delta}} \left[1 + (rr') e^{2i\tilde{\delta}} + (rr')^2 e^{4i\tilde{\delta}} + \dots \right]$$

Let

$$\tilde{\tilde{\delta}} = 2\tilde{\delta} \Rightarrow \quad \tilde{\tilde{\delta}} = 2nkd$$

and then

$$E_t = t't E_0 e^{i\tilde{\delta}} \sum_{j=0}^{\infty} (rr')^j e^{i(j\tilde{\tilde{\delta}})}$$

$$= t't E_0 e^{i\tilde{\delta}} \sum_{j=0}^{\infty} \left[rr' e^{i\tilde{\tilde{\delta}}} \right]^j$$

The geometric series can be summed. Each term has magnitude less than 1, and using

$$\sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$$

gives:

$$E_t = t't E_0 e^{i\delta} \frac{1}{1 - r'r e^{i\tilde{\tilde{\delta}}}}$$

It follows that the intensity of the transmitted wave is

$$I_t = \frac{1}{2}\epsilon v E_t E_t^* \Rightarrow \quad I_t = I_0 |t'|^2 |t|^2 \frac{1}{|1 - r'r e^{i\tilde{\tilde{\delta}}}|^2}$$

Symmetrical surface case

In the symmetrical case $|r| = |r'|$ and $|t| = |t'|$. This gives:

$$I_t = I_0 \, |t|^4 \, \frac{1}{|1 - r'r e^{i\tilde{\tilde{\delta}}}|^2}$$

Then

$$|1 - r'r e^{i\tilde{\tilde{\delta}}}|^2 = (1 - r'r e^{i\tilde{\tilde{\delta}}})(1 - r'^* r^* e^{-i\tilde{\tilde{\delta}}})$$

$$= 1 + |r'|^2 |r|^2 - (r'r e^{i\tilde{\tilde{\delta}}} + r'^* r^* e^{-i\tilde{\tilde{\delta}}})$$

$$= 1 + |r|^4 - |r|^2 \left[e^{i(\tilde{\tilde{\delta}} + \phi_r' + \phi_r)} + e^{-i(\tilde{\tilde{\delta}} + \phi_r' + \phi_r)} \right]$$

$$= 1 + |r|^4 - 2|r|^2 \cos(\tilde{\tilde{\delta}} + \phi_r' + \phi_r)$$

We then define:

$$\delta := \tilde{\tilde{\delta}} + \phi_r + \phi_r' \Rightarrow \boxed{\delta = 2nkd + \phi_r + \phi_r'}$$

giving

$$|1 - r'r e^{i\tilde{\tilde{\delta}}}|^2 = 1 + |r|^4 - 2|r|^2 \cos\delta = (1 - |r|^2)^2 + 2(1 - \underbrace{\cos\delta}_{1 - 2\sin^2 \delta/2})|r|^2$$

$$= (1 - |r|^2)^2 + 4\sin^2 \delta/2 \, |r|^2$$

Thus:

$$I_t = I_0 \frac{|t|^4}{(1 - |r|^2)^2 + 4\sin^2 \frac{\delta}{2} |r|^2} = I_0 \frac{(1 - |r|^2)^2}{\cdots}$$

$$\Rightarrow \boxed{I_t = I_0 \frac{1}{1 + \frac{4|r|^2}{(1 - |r|^2)^2} \sin^2 \delta/2}}$$

This leads to defining the coefficient of finesse

$$\boxed{F = \frac{4R}{(1-R)^2}}$$

where $R = |r|^2$ is the reflectivity. Thus

$$\boxed{I_t = I_0 \frac{1}{1 + F\sin^2(\delta/2)}}$$

<u>Exercise:</u> a) Suppose that d is fixed. Determine wavelengths that result in max transmitted intensities. Suppose $\phi_r + \phi_r' = 0$

b) Determine the maximum and describe how it depends on R.

<u>Answer:</u> a) need $\sin^2(\delta/2) = 0 \Rightarrow \delta/2 = m\pi \Rightarrow \delta = 2m\pi$

$$\Rightarrow 2nkd = 2m\pi$$

$$\Rightarrow 2n\frac{2\pi}{\lambda}d = 2m\pi$$

$$\Rightarrow \frac{4\cancel{\pi} nd}{\lambda} = 2m\cancel{\pi} \Rightarrow 2nd = m\lambda$$

$$\Rightarrow \lambda = \frac{2nd}{m} \qquad m = \pm 1, \pm 2, \ldots$$

b) $I_t = I_0 \dfrac{1}{1+F0} \Rightarrow I_t = I_0$

Does not depend on R

This motivates the definition of the coefficient of finesse

$$\boxed{F = \frac{4R}{(1-R)^2}}$$

where $R = |r|^2$ is the reflectivity. Thus

$$\boxed{I_t = I_o \frac{1}{1+F\sin^2(\delta/2)}}$$

We consider how this depends on various parameters.

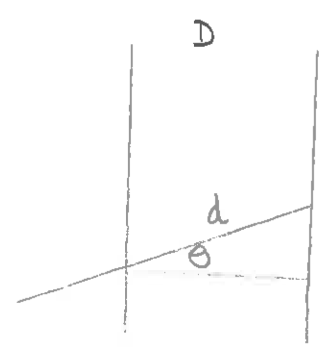
1) First, the coefficient of finesse, F, is a property of the reflecting surfaces.

2) Second, $\delta = 2nkd + \phi_r + \phi_r'$ depends on:
- wavelength of light $\lambda = 2\pi/k$
- distance traveled between successive reflections d
- reflection phase shifts. We assume that $\phi_r + \phi_r' = 0$ to simplify the analysis.

Then

$$\delta = \frac{4\pi n}{\lambda} d$$

determines the intensity.



We can see that this partly depends on the angle of incidence since

$$\frac{D}{d} = \cos\theta \quad \Rightarrow \quad d = \frac{D}{\cos\theta}$$

where D is the separation between plates.

Thus

$$\delta = \frac{4\pi n}{\lambda} \frac{D}{\cos\theta}$$

<u>Exercise:</u> For a fixed D, n, λ determine conditions that result in maximum intensity. Determine the maximum intensity.

<u>Answer:</u> We need $\sin^2(\delta/2)$ as small as possible. Thus we require

$$\sin(\delta/2) = 0 \quad \Rightarrow \quad \delta/2 = m\pi \quad \text{where} \quad m = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow \quad \frac{4\cancel{\pi} n}{\lambda} \frac{D}{\cos\theta} = 2m\cancel{\pi}$$

$$\Rightarrow \quad \boxed{2nD = m\lambda\cos\theta} \qquad m = 0 \pm 1, \pm 2, \ldots$$

Then the intensity is

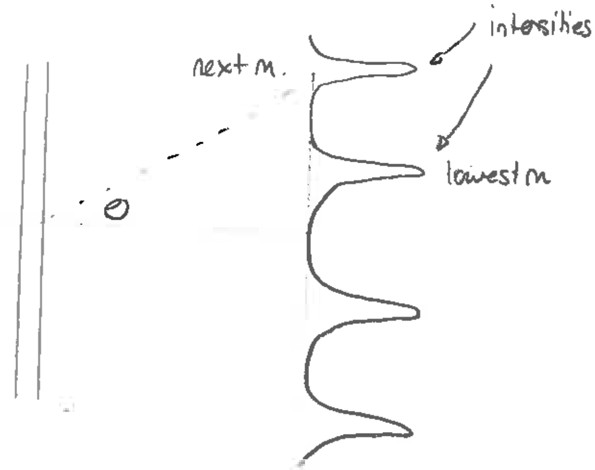
$$I_t = I_o \frac{1}{1 + F \times 0} \quad \Rightarrow \quad I_t = I_o.$$

So, for a given wavelength and separation we attain maxima at particular angles given by

$$m\cos\theta = \frac{4nD}{\lambda} \qquad m = 0, \pm 1, \pm 2, \ldots$$

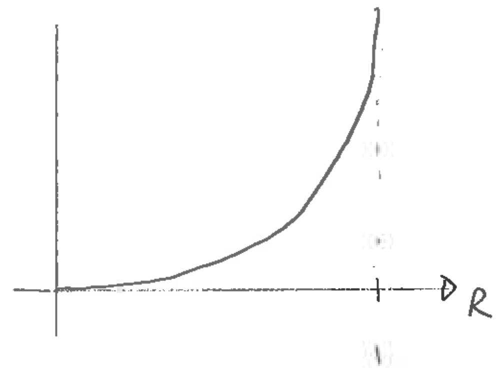
As m increases θ must also increase.

The issue now is the intensity between these maxima. As θ varies, this amounts to considering different values of δ



We want the intensity when $\delta/2 \neq m\pi$. This depends on the coefficient of finesse, which depends on R.

As $R \to 1$, $F \to \infty$. So F can be very large. In such cases I_t will vary appreciable from O only when $\delta/2 \approx m\pi$.



So we get.