HW due Weds

Velocity of Wave Packets

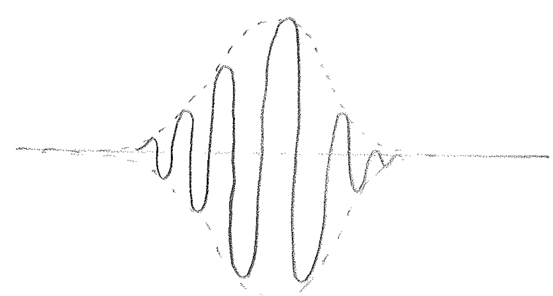
Wavepackets consist of a superposition of waves of different frequencies or wavenumbers. In general such superpositions are represented as:

$$\Psi(x,t) = \int_{-\infty}^{\infty} \tilde{\Psi}(k)\, e^{i(kx-\omega t)}\, dk$$

where Fourier analysis yields:

$$\tilde{\Psi}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ikx}\, dx$$

In various situations, these produce a packet with an envelope function. The idea behind group velocity is that it traces the progress of the envelope and ignores the progress of the more rapidly oscillating wave within the packet.



In some situations these are identical but in other situations these velocities are different.

We consider illustrating this for various cases.

Example: Two Harmonic Waves

The simplest such superposition consists of two harmonic waves with different wavenumbers. Thus consider

$$\psi_1 = A e^{i(k_1 x - \omega_1 t)}$$

$$\psi_2 = A e^{i(k_2 x - \omega_2 t)}$$

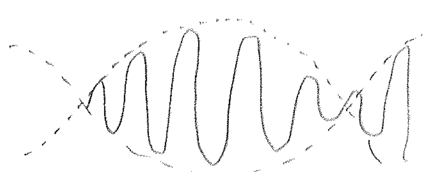
where $\omega_j = v k_j$ as usual. Then the superposition, $\psi = \psi_1 + \psi_2$ satisfies:

$$\psi(x,t) = 2A \cos\left[\frac{\Delta k x - \Delta\omega t}{2}\right] e^{i(\bar{k}x - \bar{\omega}t)}$$

where $\bar{k} = (k_1+k_2)/2$ $\quad \Delta k = k_2 - k_1$

$\bar{\omega} = (\omega_1+\omega_2)/2$ $\quad \Delta\omega = \omega_2 - \omega_1$

Again this consists of an envelope multiplied by a harmonic wave. The group velocity is the velocity with which the envelope



$$2A \cos\left[\frac{\Delta k x - \Delta\omega t}{2}\right]$$

moves. To compute this we track a maximum. For example, where $\Delta k x - \Delta\omega t = 0$. Then $x = \frac{\Delta\omega}{\Delta k} t$ gives that the group velocity is

$$v_g = \frac{\Delta\omega}{\Delta k}.$$

By contrast the velocity that appears in the dispersion relation is

$$v = \frac{\omega}{v}$$

and whenever v is dependent on k, then $\quad v_g \neq v$

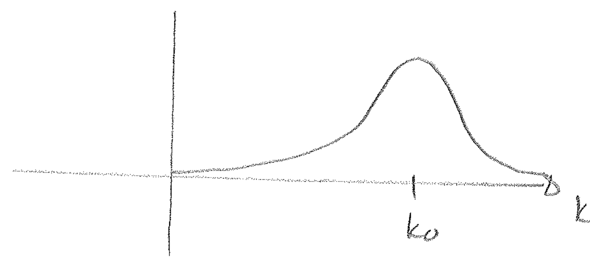
Example: Many harmonic waves.

In general one can form a superposition of infinitely many harmonic waves:

$$\Psi(x,t) = \int_{-\infty}^{\infty} \tilde{\Psi}(k)\, e^{i(kx-\omega t)}\, dk.$$

where $\omega = vk$ depends on k.
Consider the case where the wavenumbers have a Gaussian distribution centered around k_0. Set $\omega_0 = vk_0$.
Specifically

$$\tilde{\Psi}(k) = Ae^{-(k-k_0)^2/2\sigma^2}$$



Then

$$\Psi(x,t) = A\int_{-\infty}^{\infty} e^{-(k-k_0)^2/2\sigma^2} e^{i(kx-\omega t)}\, dk.$$

This can be involved, depending on whether v depends on k. Assume that v is independent of k. Then:

$$\Psi(x,t) = A\int_{-\infty}^{\infty} e^{-(k-k_0)^2/2\sigma^2}\, e^{ik(x-vt)}\, dk.$$

and performing the integral yields:

$$\Psi(x,t) = \sqrt{2\pi}\, A\sigma\, e^{ik_0(x-vt)}\, e^{-(x-vt)^2\sigma^2/2}$$

$$= \sqrt{2\pi}\, A\sigma\, e^{i(k_0x-\omega_0 t)}\, e^{-(x-vt)^2\sigma^2/2}$$

This again gives:

- a harmonic wave with frequency ω_0 and wavenumber k_0
- an envelope of the form

$$e^{-(x-vt)^2\sigma^2/2} = e^{-(x-\frac{\omega}{k}t)^2\sigma^2/2}$$

Again the envelope travels with speed $v = \omega/k$. However, this required v to be independent of k. What if v depends on k?

General Wave Packet Propagation.

In general

$$\Psi(x,t) = \int_{-\infty}^{\infty} \tilde{\Psi}(k)\, e^{i(kx-\omega t)}\, dk$$

$$= \int_{-\infty}^{\infty} \tilde{\Psi}(k)\, e^{i[kx-\omega(k)t]}\, dk.$$

Now suppose that $\tilde{\Psi}(k)$ is centered at k_0. Let $\omega_0 = \omega(k_0)$. Then define $\Delta k = k - k_0$. This gives:

$$k = \Delta k + k_0$$

$$\omega(k) = \omega(k_0) + \frac{d\omega}{dk}(k-k_0) + \dots$$

$$= \omega(k_0) + \Delta k \qquad = \omega_0 + \Delta k \frac{d\omega}{dk} + \dots$$

and thus:

$$\Psi(x,t) = \int_{-\infty}^{\infty} \tilde{\Psi}(k)\, e^{i[k_0x-\omega_0t]}\, e^{i\Delta k(x-\frac{d\omega}{dk}t)}\, d(\Delta k)$$

$$= \underbrace{e^{i(k_0x-\omega_0t)}}_{\text{harmonic wave}} \underbrace{\int_{-\infty}^{\infty} \tilde{\Psi}(k_0+\Delta k)\, e^{i\Delta k(x-\frac{d\omega}{dk}t)}\, d(\Delta k)}_{\text{envelope}}$$

We see that the waves in the envelope propagate according to $x - \frac{d\omega}{dk} t$. If this does not vary appreciably over Δk then the envelope will propagate at speed $\frac{d\omega}{dk}$. Thus we reach the fact that:

$$\boxed{\begin{array}{l}\text{The group velocity for a wavepacket is} \\ \quad v_g = \dfrac{d\omega}{dk}.\end{array}}$$

Exercise: Determine an expression for the group velocity of waves in

a) a non-dispersive medium (i.e. v does not depend on k)

b) a plasma for which

$$v = \sqrt{c^2 + \omega_0^2/k^2}$$

where ω_0 is a constant. Rewrite this in terms of ω

Answer: a)

$$\omega = vk$$

$$\frac{d\omega}{dk} = v \quad \text{since } v \text{ is constant}$$

$$\Rightarrow v_g = v$$

b)

$$\omega = kv = \sqrt{\omega_0^2 + c^2k^2}$$

$$\Rightarrow \frac{d\omega}{dk} = \frac{c^2 k}{\sqrt{\omega_0^2 + c^2k^2}} = \frac{c^2 k}{\omega}$$

So $\omega^2 = \omega_0^2 + c^2k^2$

$$\Rightarrow \quad k^2 = \frac{1}{c^2}(\omega^2 - \omega_0^2)$$

$$\Rightarrow \quad k = \frac{1}{c}\sqrt{\omega^2 - \omega_0^2}$$

and this gives:

$$V_g = \frac{c^2}{\omega}\frac{1}{c}\sqrt{\omega^2 - \omega_0^2}$$

$$= c\sqrt{1 - \omega_0^2/\omega^2}$$

Clearly $V_g \leq c$.

<u>Group velocity + index of refraction</u>

In general the index of refraction of any medium depends on the wavelength of light passing through the medium. Thus v depends on k.

<u>Exercise:</u> Determine an expression for the group velocity in terms of $n(\lambda)$, λ and derivatives of these.

<u>Answer:</u> $V_g = \frac{d\omega}{dk}$.

Now $\omega = v(k)k$ and

$$v(k) = c/n(k)$$

$$\Rightarrow \quad \omega = \frac{ck}{n(k)}$$

Thus: $$V_g = \frac{d\omega}{dk}$$

$$= \frac{c}{n(k)} + \frac{ck}{-n^2}\frac{dn}{dk}$$

$$= \frac{c}{n}\left[1 - \frac{k}{n}\frac{dn}{dk}\right]$$

$$= v\left[1 - \frac{k}{n}\frac{dn}{dk}\right].$$

Now $\frac{dn}{dk} = \frac{dn}{d\lambda}\frac{d\lambda}{dk}$. and $\lambda = \frac{2\pi}{k} \Rightarrow \frac{d\lambda}{dk} = -\frac{2\pi}{k^2}$.

So $$v_g = v\left[1 - \frac{k}{n}\left(\frac{-2\pi}{k^2}\right)\frac{dn}{d\lambda}\right]$$

$$= v\left[1 + \frac{2\pi}{k}\frac{1}{n}\frac{dn}{d\lambda}\right]$$

$$= v\left[1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right].$$

Thus we reach:

$$V_g = v\left[1 - \frac{k}{n}\frac{dn}{dk}\right]$$

$$V_g = v\left[1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right].$$

In many optical materials (e.g. glass n increases as wavelength decreases). So

$$V_g < V$$

since $\frac{dn}{d\lambda} < 1$.