Phys 473 2015

Lecture 27

Thurs: Seminar

Fri: HW 18

Fourier Transforms

The Fourier transform of any function of position $f(x)$ is a function of wavenumber

$$F(k) := \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

and the inverse transform is:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx} dk.$$

We now consider general properties of Fourier transforms.

1) Linearity

Suppose that

$$\boxed{\begin{array}{ccc} f(x) & \xrightarrow{\text{Fourier transf}} & F(k) \\ g(x) & \longrightarrow & G(k) \end{array}}$$

Then for any constants α, β

$$\boxed{\alpha f(x) + \beta g(x) \xrightarrow{\text{Fourier transf}} \alpha F(k) + \beta G(k)}$$

This linearity property follows from the linearity of integration.

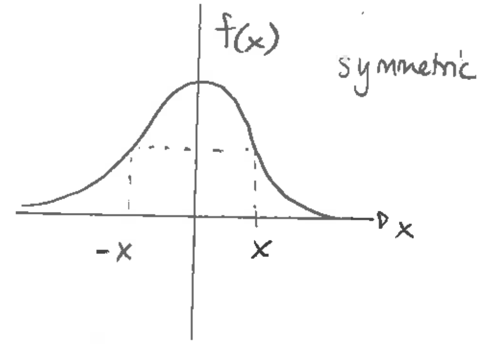
2) Symmetries.

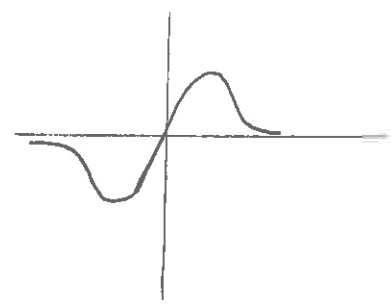
We frequently consider functions that are symmetric or antisymmetric. A function that is symmetric about $x=0$ will satisfy

$$f(-x) = f(x)$$

for all x. A function that is antisymmetric about $x=0$ will satisfy

$$f(-x) = -f(x)$$





Exercise: Suppose that $f(x)$ is real and symmetric. Show that the Fourier transform is real

Answer:
$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \int_{0}^{\infty} f(x) e^{-ikx} dx + \int_{-\infty}^{0} f(x) e^{-ikx} dx$$

Let $u=-x$ in the second integral. Then $du=-dx$

$$\int_{-\infty}^{0} f(x)\, e^{-ikx}\, dx = \int_{\infty}^{0} f(-u)\, e^{iku}\, (-du)$$

$$= \int_{0}^{\infty} f(-u)\, e^{iku}\, du$$

But $f(-u) = f(u)$ and

$$\int_{-\infty}^{0} f(x)\, e^{-ikx}\,dx = \int_{0}^{\infty} f(u)\, e^{iku}\,du$$

gives:

$$F(k) = \underbrace{\int_0^\infty f(x)\, e^{-ikx}\,dx}_{} + \underbrace{\int_0^\infty f(u)\, e^{iku}\,du}_{\text{this is complex conjugate of this}}$$

Adding $z + z^*$ always produces a real function (and symmetrical about $k=0$)

3) <u>Derivatives</u>

Given

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, e^{ikx}\,dk$$

we get

$$\frac{df}{dx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{ik\,F(k)}_{\text{must be F.T. of } \frac{df}{dx}}\, e^{ikx}\,dk$$

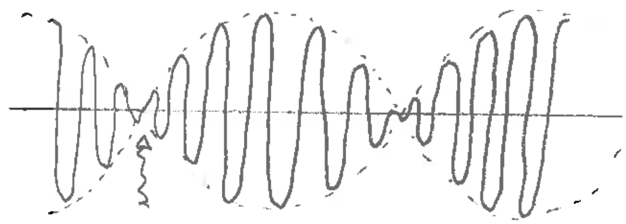
Thus

$$\boxed{\begin{array}{l} \text{If } f(x) \xrightarrow{\text{F.T.}} F(k) \text{ then} \\ \dfrac{df}{dx} \xrightarrow{\text{F.T.}} ik\,F(k) \end{array}}$$

Wave packets.

Harmonic waves are infinite in extent and therefore not a realistic physical model. How can we construct waves that are finite in extent?

We saw that a superposition of two harmonic waves with different wavenumbers or frequencies resulted in a wave whose amplitude was modulated. So at any instant, there are some areas where the amplitude is zero. If we add more of these, then we can possible increase the regions of no disturbance.

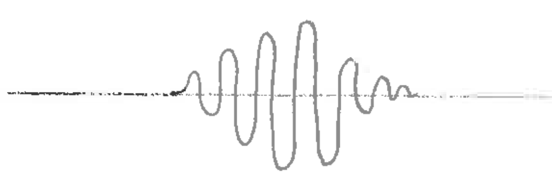


Demo Show Slides

Adding more and more harmonic waves, all with slightly different wavenumbers results in a more localized wave. Qualitatively, we see that

Greater localization in position (smaller non-zero range) $\leftrightsquigarrow$ greater range of wavenumbers

We can make this more precise by adding infinitely many such harmonic waves using the tools of Fourier analysis. If we want a localized wave, one way to construct this is



$\Psi(x,t) =$ harmonic wave with wave number k_0

$\times$ envelope function.

So

$$\boxed{\Psi(x,t) = \underbrace{f(x)}_{\text{envelope}} \underbrace{e^{ik_0 x}}_{\text{harmonic wave}}}$$

$$\Psi(x,t) = \frac{1}{2\pi}\int \tilde{\Psi}(k) e^{ikx} dk$$

gives a distribution of wavenumbers via the Fourier transform

$$\tilde{\Psi}(k) = \int_{-\infty}^{\infty} \Psi(x) e^{-ikx} dx = \int_{-\infty}^{\infty} f(x)\, e^{-i(k-k_0)x} dx$$

<u>Exercise:</u> Suppose that the envelope is given by

$$f(x) = A e^{-x^2/2\sigma^2}$$

Determine

a) the full width at half maximum of $f(x)$, Δx

b) the distribution of wavenumbers $\tilde{\Psi}(k)$

c) the full width at half maximum of the wavenumber distribution, Δk.

Note $$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Answer: a) When is $\psi(x) = \frac{1}{2}\psi_{max}$?

Hee the maximum value of ψ_{max} is A. We need x s.t

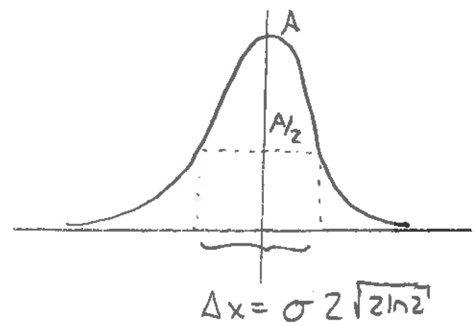
$$\psi(x) = A/2 = Ae^{-x^2/2\sigma^2}$$

$$\Rightarrow \ln\left(\tfrac{1}{2}\right) = -x^2/2\sigma^2$$

$$\Rightarrow -\ln 2 = -x^2/2\sigma^2 \Rightarrow x^2 = 2\sigma^2 \ln 2$$

$$\Rightarrow x = \pm\sigma\sqrt{2\ln 2}$$

So $\Delta x = \sigma\, 2\sqrt{2\ln 2}$



b) $$\tilde{\psi}(k) = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} e^{-i(k-k_0)x}\,dx$$

$$= \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[x^2/2\sigma^2 + i(k-k_0)x\right]}\,dx$$

Note

$$\frac{x^2}{2\sigma^2} + i(k-k_0)x = \left[\frac{x}{\sqrt{2}\sigma} + \frac{i(k-k_0)\sigma}{\sqrt{2}}\right]^2 + \frac{(k-k_0)^2\sigma^2}{2}$$

$$\Rightarrow \tilde{\psi}(k) = Ae^{-(k-k_0)^2\sigma^2/2}\int_{-\infty}^{\infty} e^{-[\ldots]^2}\,dx$$

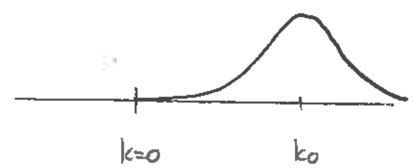
Let $u = \frac{x}{\sqrt{2}\sigma} + \frac{i(k-k_0)}{\sqrt{2}}\sigma$. Then $dx = \sqrt{2}\sigma\,du$

This gives:

$$\tilde{\psi}(k) = A e^{-(k-k_0)^2\sigma^2/2} \sqrt{2}\,\sigma \underbrace{\int_{-\infty}^{\infty} e^{-u^2} du}_{\sqrt{\pi}}$$

$$\Rightarrow \tilde{\psi}(k) = \sqrt{2\pi}\, A\sigma\, e^{-(k-k_0)^2\sigma^2/2}$$

This is another Gaussian centered at k_0



c) We need $\tilde{\psi}(k) = \frac{1}{2}\sqrt{2\pi}\, A\sigma$

$$\Rightarrow e^{-(k-k_0)^2\sigma^2/2} = 1/2$$

Thus

$$(k-k_0)^2\sigma^2/2 = \ln 2$$

$$\Rightarrow (k-k_0) = \pm\frac{\sqrt{2\ln 2}}{\sigma}$$

$$\Rightarrow k = k_0 \pm \frac{\sqrt{2\ln 2}}{\sigma}$$

So $$\Delta k = \frac{2\sqrt{2\ln 2}}{\sigma}$$

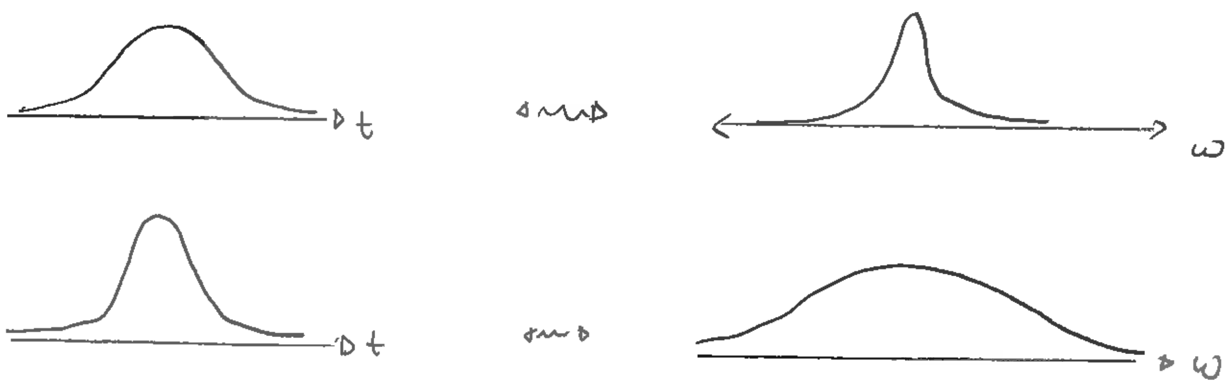
The example shows that when the FWHM is used as a measure of the spread of the wave (in position) or its constituent wavenumbers then there is an inverse relationship (depending on σ). Other measures of such ranges all show:

Spread of wave over positions decreases $\Leftrightarrow$ range of wave numbers increases

Here we can see,

$$\boxed{\Delta x \, \Delta k = 8 \ln 2}$$

A similar result applies for waves whose duration is limited to a finite interval. There will be a range of frequencies.

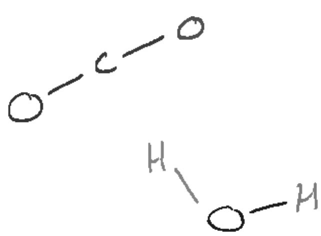


We can show that, depending on how the uncertainties are defined

$$\Delta \omega \, \Delta t > 1$$

$$\Delta k \, \Delta x > 1$$

These can be important in situations involving resonance. If we only want to excite one of many possible resonances then we cannot have a pulse of duration too short, otherwise $\Delta\omega$ will be large enough to include both frequencies. So Δt will have to be sufficiently long.



each resonate at different ω