Superpositions of many waves over a continuous range

In a given medium, harmonic waves are characterized by

- wavenumber $\vec{k}$
- amplitude
- phase

and have the form

$$\Psi(x,t) = Ae^{i(\vec{k}\cdot\vec{x} - \omega t + \phi)}$$

The angular frequency ω can be determined via the dispersion relation

$$\omega = kv$$

where $k =$ magnitude of $\vec{k}$ and the wave speed, v, might depend on k.

We have seen that superpositions of this type can be useful in general and also specifically in optical situations. We shall broaden the notion of superpositions of finitely many harmonic waves to those consisting of infinitely many waves.

For waves that propagate along one dimension, individual harmonic waves have form:

$$\Psi(x,t) = Ae^{i\phi} e^{i(kx - \omega t)}$$

$$= \tilde{A} e^{i(kx-\omega t)}$$

where $\tilde{A} = Ae^{i\phi}$ is complex.

Modifying our notation a general harmonic wave in one dimension is described by

$$\Psi(x,t) = Ae^{i(kx-\omega t)} \qquad \omega = kv$$

where A is complex and independent of x and t. Then a general superposition has the form

$$\Psi(x,t) = \text{const} \int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega t)}\, dk.$$

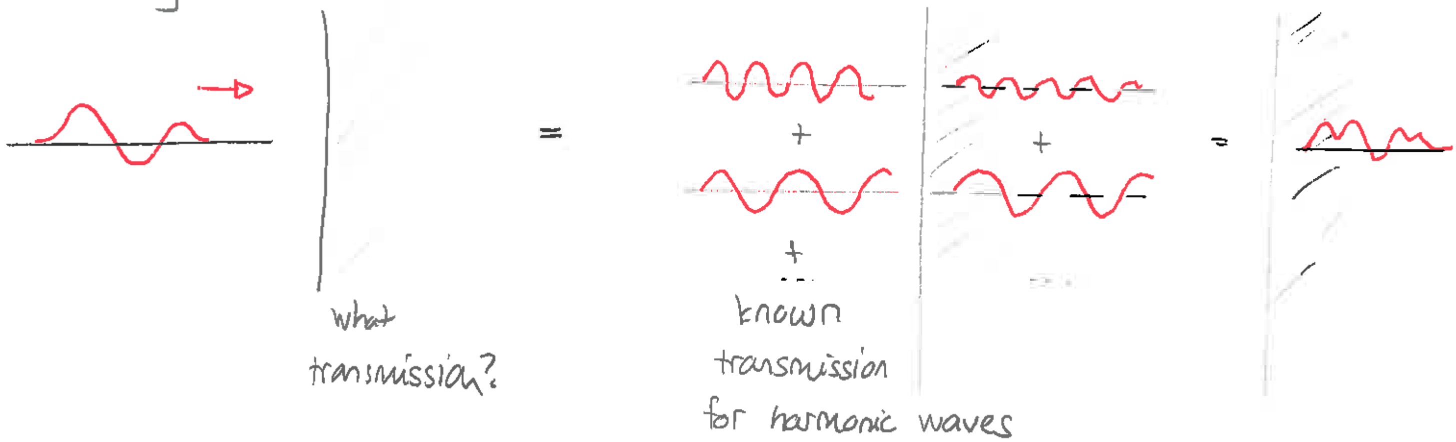
where $A(k)$ is constant and independent of t and x.

The study of such infinite integrals involving complex exponentials is called Fourier analysis. In waves this is useful for the following reasons:

1) It allows general superpositions of harmonic waves
2) It allows any wave to be represented via harmonic waves.

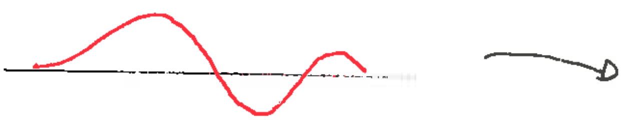
This is useful because we can often analyze the effects of optical elements + processes in terms of harmonic waves. Such effects can be recombined to ascertain the effects on a non-harmonic wave. Examples include reflection, transmission, passage through slits, etc,...

Schematically



3) It allows for a complete description of the temporal (i.e. as time passes) evolution of a wave, especially in a dispersive medium.

$\Psi(x,0)$ How to find $\Psi(x,t)$?



Decompose into a superposition by finding $\underline{\underline{A(k)}}$ s.t

$$\Psi(x,0) = \text{const} \int A(k) e^{ikx} dk$$

Then $$\Psi(x,t) = \text{const} \int A(k) e^{i(kx - \omega t)} dk$$

We will need procedures for finding the quantities $A(k)$, recombining them, etc, ...

Fourier Transforms

Given a function $f(x)$, the Fourier transform of this is:

$$\boxed{F(k) := \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx}$$

A deep mathematical theorem yields the result that this can be inverted:

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk}$$

The theorem is equivalent to:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x')\, e^{-ikx'}\, dx' \; e^{ikx}\, dk$$

$$= \int_{-\infty}^{\infty} f(x') \left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\, dk \right\} dx'$$

This can only be true for all functions, $f(x)$, if

$$\boxed{\int_{-\infty}^{\infty} e^{ik(x-x')}\, dk = 2\pi\, \delta(x-x')}$$

where $\delta(x-x')$ is the Dirac delta function. This function, by definition satisfies:

$$\delta(x-x') = \begin{cases} 0 & x \neq x' \\ \text{undef} & x = x' \end{cases}$$

and

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-x')\, dx = f(x')$$

In optics, we often transform between position (x) representations of function and wavenumber (k) representations.

Exercise Determine the Fourier transforms of

a) $f(x) = \begin{cases} 0 & |x| > a \\ A & |x| \leq a \end{cases}$

b) $f(x) = \cos(k'x)$ for $k' > 0$

Represent each graphically in terms of k.

Answer: a) $F(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$

$$= A \int_{-a}^{a} e^{ikx}\, dx = \frac{A}{ik} e^{ikx} \Big|_{-a}^{a}$$

$$\Rightarrow F(k) = \frac{A}{ik} \underbrace{\left\{ e^{ika} - e^{-ika} \right\}}_{2i \sin ka}$$

$$\Rightarrow F(k) = \frac{2A}{k} \sin ka$$

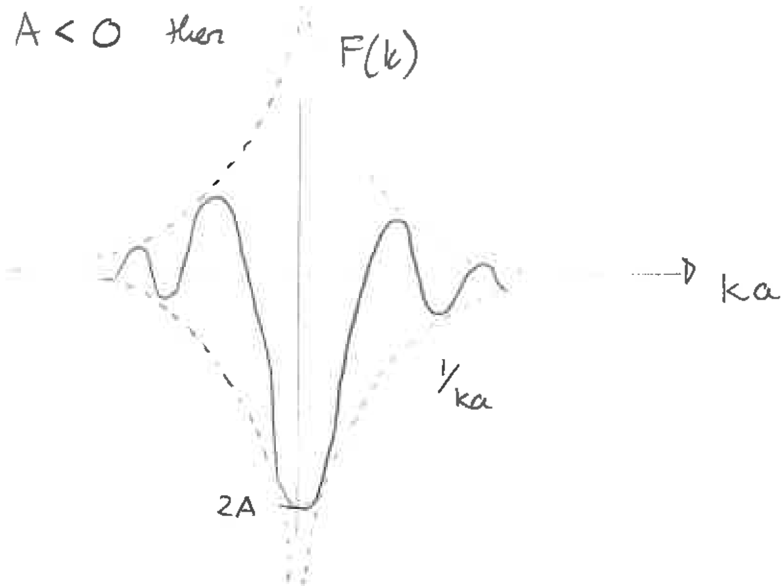
$$= 2A \frac{\sin(ka)}{ka}$$

Using the definition of the sinc function as

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

we get $F(k) = 2Aa\,\mathrm{sinc}(ka)$

If $A < 0$ then



note that as $x \to 0$

$$\frac{\sin(kx)}{kx} \to \frac{0}{0} = ??$$

By L'Hopital's rule

$$\frac{\sin(kx)}{kx} \to \frac{\frac{d}{dx}\sin(kx)}{\frac{d}{dx}(kx)}$$

$$= \frac{\cancel{k}\cos(kx)}{\cancel{k}}$$

$$\to 1$$

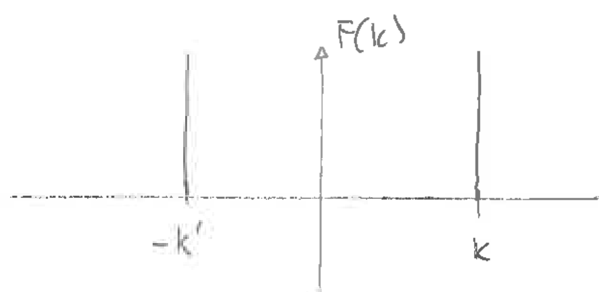
b) $$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \int_{-\infty}^{\infty} \cos(k'x)\, e^{-ikx}\, dx$$

But $$\cos(k'x) = \frac{1}{2}\left[e^{ik'x} + e^{-ik'x}\right]$$

$$\Rightarrow \quad F(k) = \frac{1}{2}\int_{-\infty}^{\infty} e^{ik'x}\, e^{-ikx}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-ik'x}\, e^{-ikx}\, dx$$

$$= \frac{2\pi}{2}\,\delta(k'-k) + \frac{2\pi}{2}\,\delta(k'+k)$$



Here, we just have two Fourier components - at k' and $-k'$.

Now suppose that we are given a waveform $\Psi(x,0)$ and need to find $\Psi(x,t)$. Then we know

$$\Psi(x,t) = \int \tilde{\Psi}(k)\, e^{i(kx-\omega t)}\, dk$$

at all times and at $t=0$

$$\Psi(x,0) = \int \tilde{\Psi}(k)\, e^{ikx}\, dk = \frac{1}{2\pi}\int 2\pi\, \tilde{\Psi}(k)\, e^{ikx}\, dk$$

By the Fourier transform:

$$2\pi\, \tilde{\Psi}(k) = \int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ikx}\, dx \Rightarrow \tilde{\Psi}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ikx}\, dx$$

So the scheme is:

Given $\Psi(x,0)$ $\Rightarrow$ Find $\tilde{\Psi}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ikx}\, dx$

$\Rightarrow$ Reconstruct

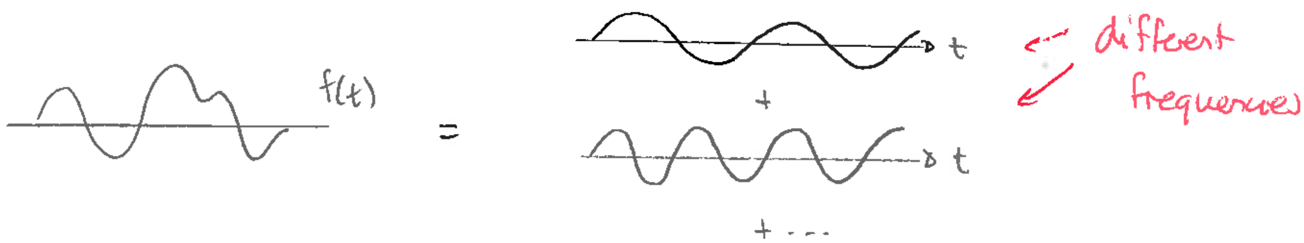
$$\Psi(x,t) = \int \tilde{\Psi}(k)\, e^{i(kx-\omega t)}\, dk.$$

The details of this depend on how $v = \omega/k$ depends on k. If v is independent of k then one can show this will always give

$$\Psi(x,t) = \Psi(x-vt,0)$$

Time domain Fourier transforms

It is frequently useful to transform a function in time domain $f(t)$ into a frequency domain function:



This will give a frequency domain representation:

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt}$$

and

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, dt.}$$

These are useful in signal processing where it might be known how signals of different frequencies are affected. An arbitrary signal can be Fourier decomposed, each component's behaviour analyzed and then the output signal reconstructed from these results.