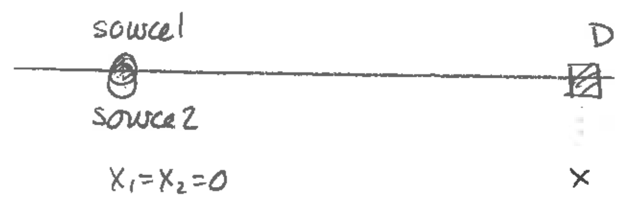
Lecture 25

Superpositions of two waves of different frequencies

We have seen the utility of a superposition of two waves of different frequencies. In the simplest situation, the sources and detector all lie along one line and this is also the direction of propagation of the wave. In a further simplification we can assume that the two sources are at the same location and at $t=0$ oscillate in phase. We can consider the case where the waves produced by the sources have the same polarization and amplitudes. Then



$$\tilde{\vec{E}}_1 = E_0 \, e^{i(k_1 x - \omega_1 t)} \, \hat{n}$$

$$\tilde{\vec{E}}_2 = E_0 \, e^{i(k_2 x - \omega_1 t)} \, \hat{n}$$

Then the complex representation of the field is:

$$\tilde{\vec{E}} = \tilde{\vec{E}}_1 + \tilde{\vec{E}}_2 = \left\{ E_0 \, e^{i(k_1 x - \omega_1 t)} + E_0 \, e^{i(k_2 x - \omega_2 t)} \right\} \hat{n}$$

and we can restrict attention to the component

$$\tilde{\vec{E}} = \left(E_0 \, e^{i\delta_1} + E_0 \, e^{i\delta_2} \right) \hat{n}$$

where $\delta_i = k_i x - \omega_i t$

So

$$\vec{E} = E_0\left(e^{i\delta_1} + e^{i\delta_2}\right)$$

$$= E_0\, e^{i(\delta_1+\delta_2)/2}\left[e^{i(\delta_1-\delta_2)/2} + e^{-i(\delta_1-\delta_2)/2}\right]$$

$$= E_0\, e^{i(\delta_1+\delta_2)/2}\; 2\cos\left(\frac{\delta_1-\delta_2}{2}\right)$$

$$\Rightarrow \quad \tilde{E} = 2E_0 \cos\left(\frac{\delta_2-\delta_1}{2}\right) e^{i(\delta_1+\delta_2)/2}$$

So the real field is:

$$E = 2E_0 \cos\left(\frac{\delta_2-\delta_1}{2}\right)\cos\left(\frac{\delta_1+\delta_2}{2}\right)$$

and with

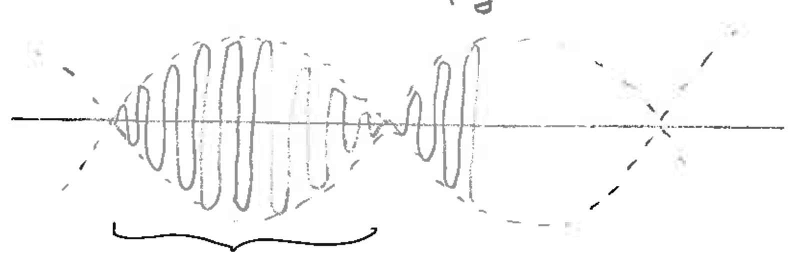
$$\bar{k} = (k_1+k_2)/2 \qquad k_{AM} = (k_2-k_1)/2$$

$$\bar{\omega} = (\omega_1+\omega_2)/2 \qquad \omega_{AM} = (\omega_2-\omega_1)/2$$

we get

$$E = 2E_0 \underbrace{\cos\left[k_{AM}x - \omega_{AM}t\right]}_{\substack{\text{fluctuates slowly} \\ = \text{envelope}}} \underbrace{\cos\left[\bar{k}x - \bar{\omega}t\right]}_{\substack{\text{fluctuates rapidly} \\ = \text{signal}}}$$

multiply



"packet"

We can see that the wave now consists of a train of "packets" and these are described by

$$\cos\left[k_{AM}x - \omega_{AM}t\right]$$

Group velocity

If a signal is to be sent using the packets then it will propagate at the speed given by the packets.

Exercise: a) Consider the crest of a packet Determine an expression for $k_{AM}x - \omega_{AM}t$ for a crest of the packet. Use this to determine an expression for the location of the crest of the packet as a function of time.

b) Determine an expression for the velocity of the packet

Answer: a) Any of $k_{AM}x - \omega_{AM}t = n\pi$ where n is an integer work

$$\Rightarrow \quad x = \frac{\omega_{AM}}{k_{AM}}t + \frac{n\pi}{k_{AM}}$$

b) By inspection the velocity of the packet is $\omega_{AM}/k_{AM} = \frac{\Delta\omega}{\Delta k}$

The velocity with which the packet propagates is called the group velocity:

$$\boxed{V_g = \text{velocity with which packet propagates}}$$

Here we find that

For any superposition of two waves

$$V_g = \frac{\Delta\omega}{\Delta k}.$$

It is always true that the wavelength and frequency of harmonic waves are related by

$$V = \omega/k. \Rightarrow \omega = Vk.$$

where V is properly called the phase velocity and is sometimes denoted V_{ph}. What is not always true is that the phase velocity is independent of ω.

Exercise: For light in a vacuum the phase velocity is c and is independent of ω.
Determine an expression for the group velocity of light waves in a vacuum.

Answer: Here $\omega = ck \Rightarrow V_{ph} = \omega/k = c$

Then $\Delta\omega = c\,\Delta k \Rightarrow V_g = \frac{\Delta\omega}{\Delta k} = c$

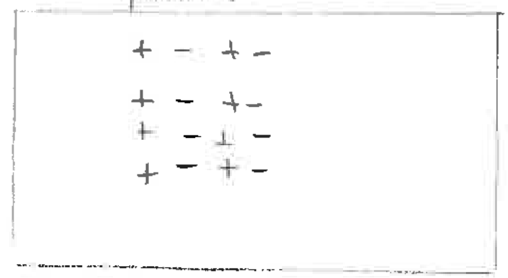
So for waves in a vacuum

i) the phase velocity is independent of frequency
ii) the group velocity equals the phase velocity.

This is not always true. If waves travel through a plasma (mixture of positively + negatively charged particles) we find that

$$\omega^2 = \omega_0^2 + c^2 k^2$$

plasma



Exercise: Suppose that $\omega_1 = 2\omega_0$ and $\omega_2 = 3\omega_0$. for waves in a plasma. Determine:

a) the phase velocity for each

b) the group velocity

Answer: a) $v = \omega/k$ and $k = \frac{1}{c}\sqrt{\omega^2 - \omega_0^2}$

$$\Rightarrow \quad v = \frac{c\,\omega}{\sqrt{\omega^2 - \omega_0^2}} = \frac{c}{\sqrt{1 - \omega_0^2/\omega^2}}$$

with $\omega_1 = 2\omega_0$ $\quad v = \frac{c}{\sqrt{1 - 1/4}} = \sqrt{\frac{4}{3}}\, c$

$\omega_2 = 3\omega_0$ $\quad v = \frac{c}{\sqrt{1 - 1/9}} = \sqrt{\frac{9}{8}}\, c$

These are both larger than c but are different.

b) We need $V_g = \frac{\Delta\omega}{\Delta k} = \frac{\omega_0}{\Delta k}$

Now $\quad k_1 = \frac{1}{c}\sqrt{3\omega_0^2} = \frac{\sqrt{3}\,\omega_0}{c}$

$k_2 = \frac{1}{c}\sqrt{8\omega_0^2} = \frac{\sqrt{8}\,\omega_0}{c}$

So $\Delta k = (\sqrt{8} - \sqrt{3})\, \omega_0 / c$

$$\Rightarrow \qquad V_g = \frac{c\, \cancel{\omega_0}}{\cancel{\omega_0}(\sqrt{8}-\sqrt{3})} = \frac{c}{1.09}$$

and this is less than c.

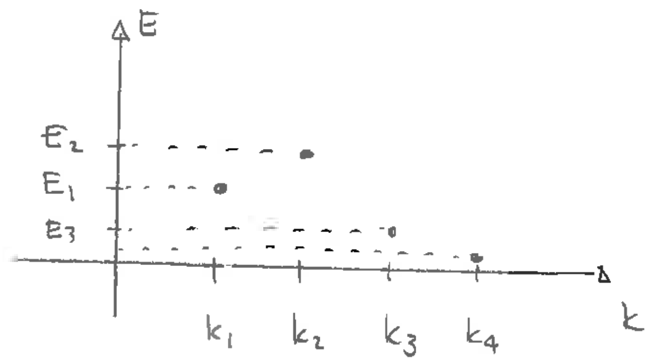
Superpositions of multiple waves with multiple frequencies

In general we can provide a relationship between wave frequency and wave number via $\omega = vk$. We only need to specify k to describe the wave (with the usual one dimensional constraints). We can now consider general superpositions:

$$\tilde{E} = E_1 e^{i(k_1 x - \omega_1 t)} + E_2 e^{i(k_2 x - \omega_2 t)} + E_3 e^{i(k_3 x - \omega_3 t)} + \dots$$

$$= \sum E_j e^{i(k_j x - \omega_j t)}$$

where $\omega_j = v k_j$. Conceptually we can represent this via graphing the amplitudes E_j as functions of wavenumbers.

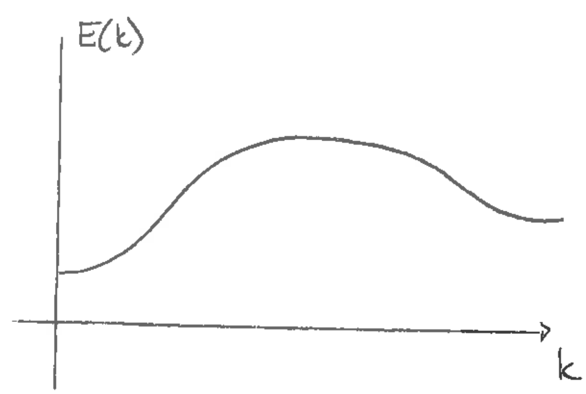


Instead of adding discrete sums, we will consider extensions to superpositions over continuous distributions. These have the form

$$\tilde{E} = \text{const} \int E(k) e^{i(kx - \omega t)} dk$$

$$\omega = kv$$

for a suitable weighting constant. We will represent the distribution $E(k)$ via a continuous graph.



Issues arising from this are:

Given $\tilde{E}(x)$ at some initial moment, can we find $E(t)$?

Given $\dot{E}(x)$ " " " " , can we find $E(x,t)$?

The subject of Fourier transforms addresses this