Weds: HW due

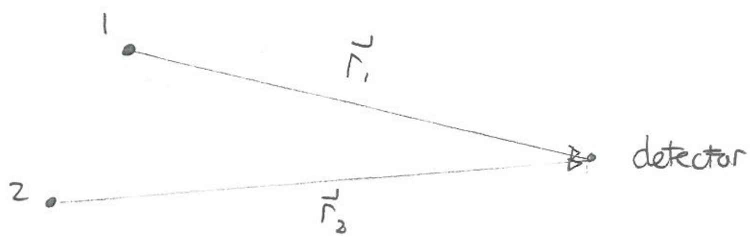
Thurs: Seminar

~~Wed~~ Fri: HW due

Superpositions of Waves of Different Frequencies.

So far we have mostly considered addition of waves from two sources that radiate at the same frequency. We dismissed the situation where the sources have distinct frequencies on the grounds that over infinite times the time average would yield no interference. We now investigate this in more detail.

Consider two sources



Then using the complex representation, the fields produced by each source at the detector is

$$\vec{E}_1 = \vec{E}_{01} e^{i(\vec{k}_1 \cdot \vec{r}_1 - \omega_1 t + \phi_1)}$$

$$\vec{E}_2 = \vec{E}_{02} e^{i(\vec{k}_2 \cdot \vec{r}_2 - \omega_2 t + \phi_2)}$$

Thus the combined field is:

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

We assume that the two polarizations are parallel, along an axis $\hat{n}$ and that the sources radiate with equal electric fields. Then

$$\vec{E}_{01} = E_0 \hat{n}$$
$$\vec{E}_{02} = E_0 \hat{n}$$

Thus the complex representation of the field is:

$$\tilde{\vec{E}} = \left[E_0 e^{i\delta_1} + E_0 e^{i\delta_2}\right]\hat{n}$$

where for convenience we have defined

$$\boxed{\delta_i = \vec{k}_i \cdot \vec{r}_i - \omega_i t + \phi_i}$$

Then the irradiance is determined by:

Extract real representation of field from complex rep

$$\vec{E} = Re(\tilde{\vec{E}})$$

Time dependent irradiance is

$$I = \epsilon v \vec{E} \cdot \vec{E}$$

Extract time independent irradiance by

$$\frac{1}{T}\int_0^T I(t)\,dt$$

where T is shorter than detector resolution but longer than typical period of wave oscillation

We note that

$$\vec{E} = Re(\tilde{\vec{E}}_c) = \left[E_0 \cos\delta_1 + E_0 \cos\delta_2\right]\hat{n}$$

and this would, in principle let us determine $\vec{E} \cdot \vec{E}$. However we propose an alternative method that can be generalized easily to a situation for many sources.

Exercise: a) Rewrite the expression for $\tilde{\vec{E}}$ in terms of $\delta_2-\delta_1$ and $\delta_1+\delta_2$

b) Rewrite the result of a) with trig functions replacing complex exponentials. Extract $\vec{E}$

Answer: a)

$$\delta_1 = \frac{(\delta_1+\delta_2)+(\delta_2-\delta_1)}{2}$$

$$\delta_2 = \frac{(\delta_1+\delta_2)-(\delta_2-\delta_1)}{2}$$

$$\Rightarrow \tilde{\vec{E}} = E_0\hat{n}\left\{e^{i(\delta_1+\delta_2)/2}e^{-i(\delta_2-\delta_1)/2} + e^{i(\delta_1+\delta_2)/2}e^{+i(\delta_2-\delta_1)/2}\right\}.$$

$$= E_0\hat{n}\, e^{i(\delta_1+\delta_2)/2}\left\{e^{-i(\delta_2-\delta_1)/2} + e^{+i(\delta_2-\delta_1)/2}\right\}$$

b) $$\tilde{\vec{E}} = E_0\hat{n}\left\{\cos\left[(\delta_1+\delta_2)/2\right] + i\sin\left[(\delta_1+\delta_2)/2\right]\right\} 2\cos\left(\frac{\delta_2-\delta_1}{2}\right)$$

$$\Rightarrow \vec{E} = 2E_0\cos\left(\frac{\delta_2-\delta_1}{2}\right)\cos\left(\frac{\delta_1+\delta_2}{2}\right)\hat{n}$$

Now

$$\frac{\delta_1+\delta_2}{2} = \frac{\vec{k}_1\cdot\vec{r}_1+\vec{k}_2\cdot\vec{r}_2}{2} - \frac{\omega_1+\omega_2}{2}t + \frac{\phi_1+\phi_2}{2}$$

$$\frac{\delta_2-\delta_1}{2} = \frac{\vec{k}_2\cdot\vec{r}_2-\vec{k}_1\cdot\vec{r}_1}{2} - \frac{\omega_2-\omega_1 t}{2} + \frac{\phi_2-\phi_1}{2}.$$

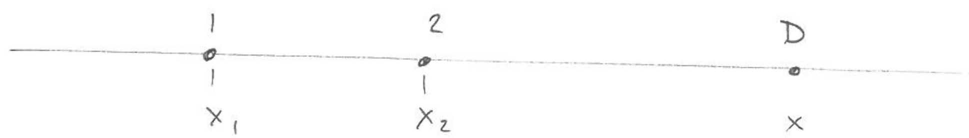
In general the field depends on sums and differences of these parameters.

This general case carries extraneous complication. We simplify it as follows:

i) assume $\phi_1 = \phi_2 = 0$

ii) assume the sources + detector lie along the x-axis

iii) assume that the wave propagates along x



Then
$$\vec{r}_1 = (x - x_1)\,\hat{x}$$
$$\vec{r}_2 = (x - x_2)\,\hat{x}$$
$$\vec{k}_i = k_i\,\hat{x}$$

gives:

$$\frac{\delta_1+\delta_2}{2} = \frac{k_1(x-x_1) + k_2(x-x_2)}{2} - \frac{\omega_1+\omega_2}{2}t = \frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t - \frac{k_1x_1+k_2x_2}{2}$$

$$\frac{\delta_2-\delta_1}{2} = \frac{k_2(x-x_2) - k_1(x-x_1)}{2} - \frac{\omega_2-\omega_1}{2}t = \frac{k_2-k_1}{2}x - \frac{\omega_2-\omega_1}{2}t - \frac{k_2x_2-k_1x_1}{2}$$

Thus we define

mean frequency $\bar{\omega} := (\omega_1+\omega_2)/2$

mean wavenumber $\bar{k} := (k_1+k_2)/2$

modulation frequency $\omega_{AM} := (\omega_2-\omega_1)/2$

modulation wavenumber $k_{AM} := (k_2-k_1)/2$

phases $\alpha_+ = (k_1x_1+k_2x_2)/2$

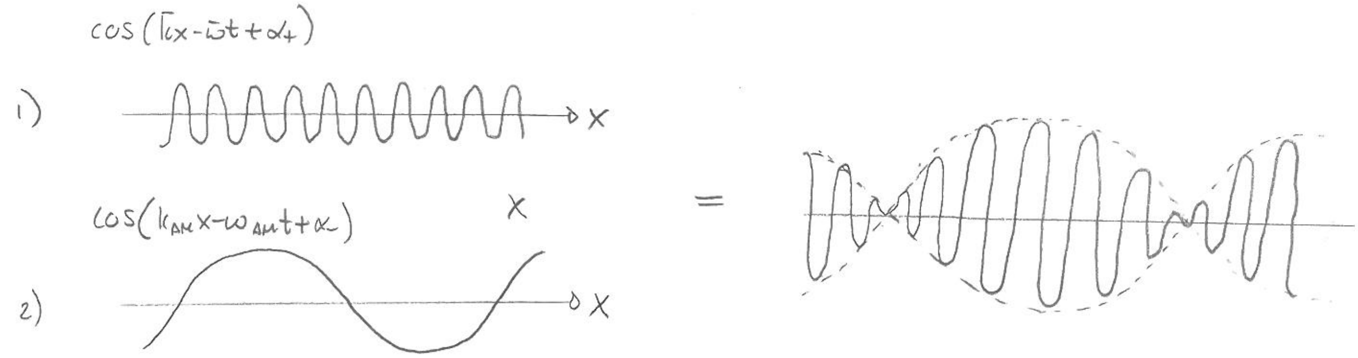
$\alpha_- = (k_2x_2-k_1x_1)/2$

Then

$$\boxed{\bar{E} = 2E_0 \cos\left[k_{AM}x - \omega_{AM}t - \alpha_-\right]\cos\left[\bar{k}x - \bar{\omega}t - \alpha_+\right]\hat{n}}$$

In general we consider cases where $\omega_{AM} < \bar{\omega}$ and $k_{AM} < \bar{k}$. Then the field is described in terms of two factors:

1) a "carrier" wave with wavenumber $\bar{k}$ and frequency $\bar{\omega}$
2) a modulating amplitude with wavenumber k_{AM} and frequency ω_{AM}

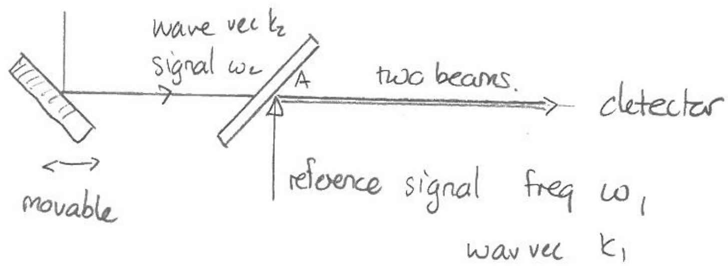


This is how AM transmission works. The amplitude carries the message to be transmitted

In classical waves this phenomena is also called a "beat"!

Intensity

We now consider how this can manifest itself in terms of intensity. If $\omega_{AM} = (\omega_2 - \omega_1)/2$ is relatively low (compared to typical optical frequencies) then we might be able to observe the fluctuations in $\vec{E}^2$. This is used in heterodyne detection and also coherent Doppler Lidar. In heterodyne detection, a beam splitter combines two beams



We can set up $x_1 = x_2 = 0$ at the point where the two beams emerge after the beam splitter. We need an expression for the intensity at the detector located at position x. We find that

> If the time over which the detector averages, T, satisfies $T \gg \frac{1}{\bar{\omega}}, \frac{1}{\omega_1}, \frac{1}{\omega_2}$, $T \ll \frac{1}{\Delta\omega}$ then at time t the intensity is:
>
> $$I = I_{ref} + I_{sig} + \epsilon v\, E_{ref} E_{sig} \cos(\Delta\omega t + \alpha)$$
>
> where $\Delta\omega = \omega_1 - \omega_2$

Proof: From before:

$$\vec{E} = \vec{E}_{ref}\, e^{i(k_1 x - \omega_1 t)} + \vec{E}_{sig}\, e^{i(k_2 x - \omega_2 t)}$$

$$\Rightarrow \vec{E} = \vec{E}_{ref} \cos(k_1 x - \omega_1 t) + \vec{E}_{sig} \cos(k_2 x - \omega_2 t)$$

So the instantaneous irradiance is

$$I = \epsilon v \vec{E}^2$$

$$= \epsilon v \left\{ E_{ref}^2 \cos^2(k_1 x - \omega_1 t) + E_{sig}^2 \cos(k_2 x - \omega_2 t) \right.$$

$$\left. + 2 \vec{E}_{ref} \cdot \vec{E}_{sig} \cos(k_1 x - \omega_1 t) \cos(k_2 x - \omega_2 t) \right\}$$

Then the time average gives three terms:

$$\frac{1}{T} \int_t^{t+T} \cos^2(k_1 x - \omega_1 \tau) \, d\tau \rightarrow \frac{1}{2} \quad \text{if} \quad \omega_1 T \gg 1$$

$$\frac{1}{T} \int_t^{t+T} \cos^2(k_2 x - \omega_2 \tau) \, d\tau \rightarrow \frac{1}{2} \quad \text{if} \quad \omega_2 T \gg 1$$

$$\frac{1}{T} \int_t^{t+T} \cos(k_1 x - \omega_1 \tau) \cos(k_2 x - \omega_2 \tau) \, d\tau$$

Thus

$$I = \underbrace{\frac{1}{2} \epsilon v E_{ref}^2}_{I_{ref}} + \underbrace{\frac{1}{2} \epsilon v E_{sig}^2}_{I_{sig}} + 2 \vec{E}_{ref} \cdot \vec{E}_{sig} \frac{1}{T} \int_t^{t+T} \cos(k_1 x - \omega_1 \tau) \cos(k_2 x - \omega_2 \tau) \, d\tau$$

We focus on the last term. Then $\cos A \cos B = \dfrac{\cos(A+B) + \cos(A-B)}{2}$ gives

$$\int_t^{t+T} \cos(\ldots) \cos(\ldots) \, d\tau = \frac{1}{2} \int_t^{t+T} \cos\left((k_1 + k_2) x - (\omega_1 + \omega_2) \tau\right) d\tau$$

$$+ \frac{1}{2} \int_t^{t+T} \cos(\Delta k x - \Delta \omega \tau) \, d\tau$$

where $\Delta k = k_1 - k_2$, $\Delta \omega = \omega_1 - \omega_2$

The first integral satisfies $\bar{\omega} T \gg 1$ and reduces to 0

The second gives: $$\frac{-1}{\Delta\omega} \sin(\Delta k\, x - \Delta\omega\, \tau) \Big|_t^{t+T}$$

$$= -\frac{1}{\Delta\omega}\left[\sin(\Delta k x - \Delta\omega t - \Delta\omega T) - \sin(\Delta k x - \Delta\omega t)\right]$$

Now $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$ implies that the second gives.

$$-\frac{2}{\Delta\omega} \cos\left[\Delta k x - \Delta\omega t - \frac{\Delta\omega T}{2}\right] \sin\left(-\frac{\Delta\omega T}{2}\right)$$

Thus

$$I = I_{ref} + I_{sig} + ev2\vec{E}_{ref}\cdot\vec{E}_{sig} \frac{\cancel{2}}{\cancel{2}\Delta\omega T} \sin\left(\frac{\Delta\omega T}{2}\right) \cos\left[\Delta k x - \Delta\omega t - \frac{\Delta\omega T}{2}\right]$$

If $\Delta\omega T \ll 1$ then $\sin\frac{\Delta\omega T}{2} \approx \frac{\Delta\omega T}{2}$ and so

$$I = I_{ref} + I_{sig} + ev\,\vec{E}_{ref}\cdot\vec{E}_{sig} \cos\left(\Delta k x - \Delta\omega t - \frac{\Delta\omega T}{2}\right)$$

$$= I_{ref} + I_{sig} + ev\,\vec{E}_{ref}\cdot\vec{E}_{sig} \cos(\Delta\omega t + \alpha)$$

where $\alpha = -\Delta k\, x + \Delta\omega T/2$. ☒

Thus we will note that with signals of two different frequencies combined in this way, the intensity oscillates with time. The angular frequency of oscillation is $\Delta\omega$.

In Doppler Lidar a reflection off a moving source generates a frequency shift

$$f_0 \Rightarrow f' = f_0 \sqrt{\frac{1+v/c}{1-v/c}}$$

that depends on velocity. The receiver can measure Δf and use the known signal f_0 to obtain f' and then v.

Exercise: Given Δf and f_0, determine an expression for

a) v/c

b) v.

Answer:

$$f' = \Delta f + f_0 = f_0 \sqrt{\frac{1+v/c}{1-v/c}}$$

$$\frac{\Delta f + f_0}{f_0} = \sqrt{} \quad \Rightarrow \quad \left(1 + \frac{\Delta f}{f_0}\right)^2 = \frac{1+v/c}{1-v/c}.$$

Now $$\frac{1+a}{1-a} = b \quad \Rightarrow \quad (1+a) = (1-a)b$$

$$\Rightarrow \quad 1 - b = -a(b+1)$$

$$\Rightarrow \quad a = \frac{b-1}{b+1}$$

So $$\frac{v}{c} = \frac{\left(1+\frac{\Delta f}{f_0}\right)^2 - 1}{\left(1+\frac{\Delta f}{f_0}\right)^2 + 1} = \frac{\frac{\Delta f}{f_0}\left(2 + \frac{\Delta f}{f_0}\right)}{\left(1+\frac{\Delta f}{f_0}\right)^2 + 1}$$

$$\Rightarrow v = c \; \frac{\frac{\Delta f}{f_0}\left(2 + \frac{\Delta f}{f_0}\right)}{1 + \left(1+\frac{\Delta f}{f_0}\right)^2}$$