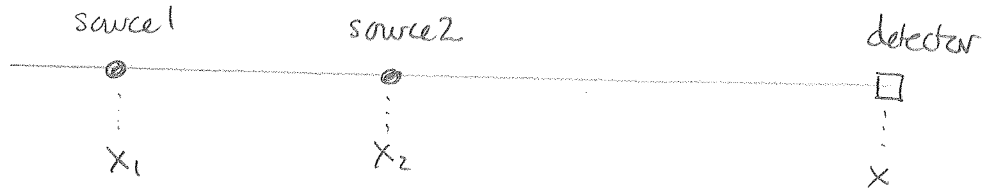
Phys 473 2015

Lecture 22

Superposition of waves

We considered the following arrangement for superposition of waves



wavenumber	$k_1\hat{x}$	$k_2\hat{x}$
frequency	ω_1	ω_2
phase	ϕ_1	ϕ_2

Then the field at the detector is described by:

$$\vec{E} = \vec{E}_{01} e^{i(k_1(x-x_1)-\omega_1 t+\phi_1)} + \vec{E}_{02} e^{i(k_2(x-x_2)-\omega_2 t+\phi_2)}$$

and the time averaged intensity is:

If $\omega_1 \neq \omega_2$ then

$$I = I_1 + I_2$$

where $I_i = \frac{1}{2}\epsilon v E_{0i}^2$ is the intensity due to i alone.

If $\omega_1 = \omega_2$ then

$$I = I_1 + I_2 + \epsilon v \vec{E}_{01}\cdot\vec{E}_{02}\cos\delta$$

Here

$$\boxed{\delta := k(x_1 - x_2) + \phi_2 - \phi_1}$$

is the phase difference between the waves. We see that the intensity depends on the relative polarization of the waves and the phase difference.

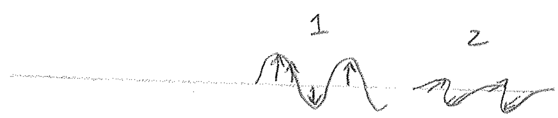
Polarization considerations

If the polarizations of the waves are perpendicular then $\vec{E}_{01} \cdot \vec{E}_{02} = 0$ and we have

$$I = I_1 + I_2$$

In this case it is as though the two waves exist independently of each other. There is no interference.

Now suppose that the polarizations are parallel.



Exercise: Suppose that $\vec{E}_{01}$ and $\vec{E}_{02}$ are parallel. Determine

a) an expression for I in terms of I_1, I_2 and δ only.

b) the maximum value for I and phases at which this occurs

c) the minimum " " I " " " " " "

Answer: a) $\vec{E}_{01} \cdot \vec{E}_{02} = E_{01} E_{02}$

$\frac{1}{2} \epsilon v E_{01}^2 = I_1 \Rightarrow E_{01} = \sqrt{2 I_1 / \epsilon v}$

So $\vec{E}_{01} \cdot \vec{E}_{02} = \sqrt{\frac{2I_1}{\epsilon v}} \sqrt{\frac{2I_2}{\epsilon v}}$

$\Rightarrow \quad I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\delta .$

b) max when $\cos\delta = 1$

$\Rightarrow I = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$

here $\delta = 2n\pi \quad n = 0, 1, 2, \ldots$

c) min when $\cos\delta = -1$

$I = I_1 + I_2 - 2\sqrt{I_1 I_2} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$

occurs where $\delta = (2n+1)\pi \quad n = 0, 1, 2, \ldots$

This is plotted in the general case.

The extreme case is attained when

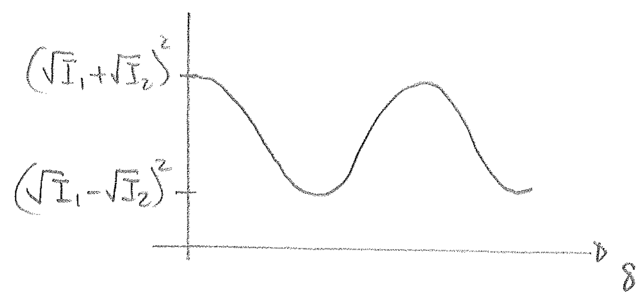
$I_1 = I_2$

and here the max intensity is:

$I = 4 I_1$

and the minimum intensity is

$I = 0$.



These fluctuations in intensity are examples of interference. The two extremes illustrated here are called constructive and destructive interference respectively.

Visibility

Interference is quantified by fluctuations in intensity. One measure of the extent of such interference is the contrast between greater + smaller intensities as δ varies. To this end

Let I_{max} be the maximum intensity as δ varies
Let I_{min} " " minimum " " δ varies.

Then the visibility (or "fringe" visibility) is

$$\boxed{V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}.}$$

Exercise: Determine an expression for the visibility for the previous case;

Answer:

$$I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2 = I_1 + I_2 + 2\sqrt{I_1 I_2}$$

$$I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2 = I_1 + I_2 - 2\sqrt{I_1 I_2}$$

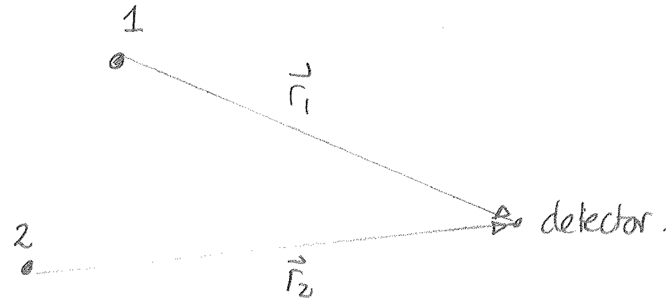
$$\Rightarrow \quad V = \frac{4\sqrt{I_1 I_2}}{2(I_1 + I_2)} = \frac{2\sqrt{I_1 I_2}}{(I_1 + I_2)}$$

When $I_1 = I_2$ then $V = 1$

When $I_1 = 0$ or $I_2 = 0$ then $V = 0$.

General superposition

In general the sources, detector and propagation direction might not all be aligned



Let $\vec{r}_1$ and $\vec{r}_2$ be displacement vectors from sources to the detector. Then the field at the detector is:

$$\vec{E} = \vec{E}_{01}\, e^{i(\vec{k}_1\cdot\vec{r}_1 - \omega_1 t + \phi_1)} + \vec{E}_{02}\, e^{i(\vec{k}_2\cdot\vec{r}_2 - \omega_2 t + \phi_1)}$$

Again computing the intensity gives:

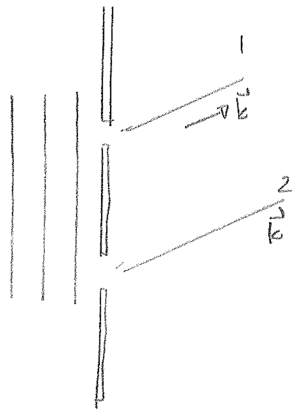
If	$\omega_1 \neq \omega_2$	then	$I = I_1 + I_2$
If	$\omega_1 = \omega_2$	then	$I = I_1 + I_2 + 2\vec{E}_{01}\cdot\vec{E}_{02}\cos\delta$

where I_j is the intensity due to wave j alone and the phase difference is

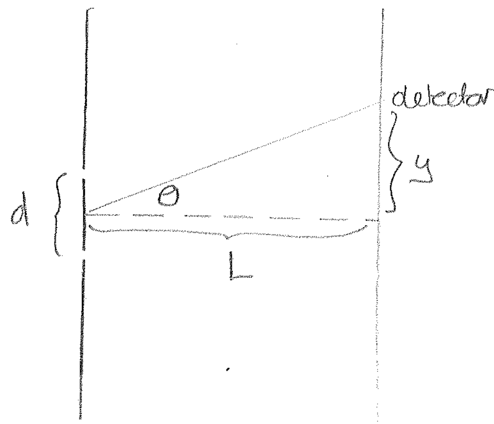
$$\boxed{\delta = \vec{k}_2\cdot\vec{r}_2 - \vec{k}_1\cdot\vec{r}_1 + \phi_2 - \phi_1}$$

Double slit experiment

In the double slit experiment a single source illuminates two closely spaced slits in an opaque barrier. The typical set-up assumes that the incident waves are plane waves with wave vector perpendicular to the barrier. Then the detector is assumed to be very distant compared to the gap between the slits. We assume:



1) the electric fields have the same amplitude and polarization at each opening.
2) the fields are in phase at each opening.



Then the phase difference between 1 and 2 is:

$$\delta = \vec{k}\cdot\vec{r}_2 - \vec{k}\cdot\vec{r}_1$$

$$= k(r_2 - r_1)$$

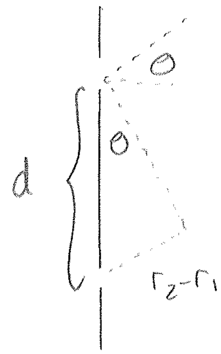
We now express this in terms of the separation between the slits, d, and the angle θ from the midpoint to the detector.

<u>Exercise:</u> a) Determine an expression for $r_2 - r_1$ in terms of d, θ.

b) Suppose that $\vec{E}_{01} = \vec{E}_{02}$. Determine an expression for the intensity at any angle θ.

c) Determine conditions for maximum and minimum irradiance.

Answers: a) Using geometry + trigonometry



$$\frac{r_2-r_1}{d} = \sin\theta$$

$$\Rightarrow r_2-r_1 = d\sin\theta$$

b) Here $I_2 = I_1$ and we get

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$$

$$= I_1 + I_1 + 2I_1\cos\delta$$

$$= 2I_1(1+\cos\delta)$$

where $\delta = k(r_2-r_1) = kd\sin\theta$

$$\Rightarrow I = 2I_1\left[1+\cos(kd\sin\theta)\right]$$

Now $\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \Rightarrow \cos\alpha + 1 = 2\cos^2\alpha/2.$

Thus

$$I = 4I_1\cos^2\left[\frac{kd\sin\theta}{2}\right]$$

c) This is max when $\frac{kd}{2}\sin\theta = n\pi$ $\quad n = 0, \pm1, \pm2, \ldots$

$$\Rightarrow \frac{\cancel{2}\cancel{\pi}}{\lambda}\frac{d}{\cancel{2}}\sin\theta = n\cancel{\pi} \Rightarrow d\sin\theta = n\lambda.$$

This is min when $\frac{kd}{\cancel{2}}\sin\theta = \frac{2n+1}{\cancel{2}}\pi$

$$\Rightarrow d\sin\theta = \frac{2n+1}{2}\lambda \qquad n = 0, \pm1, \pm2, \pm3, \ldots$$