Phys 473 2015

Lecture 21

HW due Weds

Electromagnetic Waves

In classical physics light is described in terms of electromagnetic waves. We generally consider harmonic wave solutions, in which the wave is completely described by an electric field

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t + \phi)}$$

where $\vec{E}_0$ is independent of $\vec{r}, t$

$\vec{k}$ is the wave vector

ω is the angular frequency

ϕ is the phase

Then the dispersion relation gives:

$$\omega = kv$$

where v is the speed of light in the medium. The magnetic field is determined via applying Maxwell's equations and this gives

$$\vec{B} = \frac{1}{\omega} \vec{k} \times \vec{E}$$

The power per unit area transported by this wave is determined by the Poynting vector

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

It follows that for such electromagnetic waves

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \left(\frac{1}{\omega} \vec{k} \times \vec{E}\right)$$

$$= \frac{1}{\mu_0} \frac{\vec{k}}{\omega} (\vec{E} \cdot \vec{E})$$

and this has magnitude

$$S = \frac{k}{\mu_0 \omega} E^2 = \frac{\cancel{\omega}}{v \mu_0 \cancel{\omega}} E^2$$

and with $v = \frac{1}{\sqrt{\epsilon \mu_0}} \Rightarrow \frac{1}{\mu_0} = \epsilon v^2$ we get that the <u>time-dependent irradiance</u> is:

$$S = \epsilon v E^2 = \epsilon v E_0^2 \cos^2(2\vec{k}\cdot\vec{r} - 2\omega t + 2\phi)$$

In general we detect a time independent irradiance

$$\boxed{I := \langle S \rangle}$$

where the time average of any function is

$$\langle f \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(t)\, dt$$

and for a monochromatic plane wave we can easily show that

$$I = \frac{1}{2} \epsilon v E_0^2$$

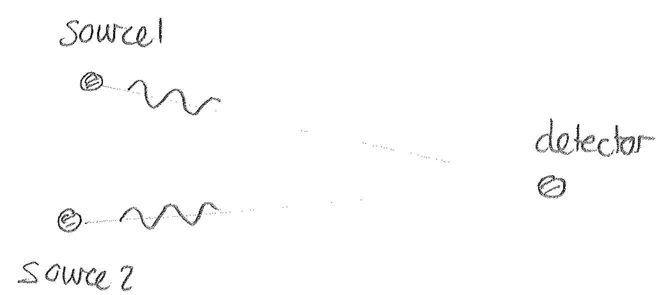
Superposition of Waves

The wave equation allows for the existence of superposition of solutions. So if $\vec{E}_1(\vec{r},t)$ and $\vec{E}_2(\vec{r},t)$ are any two solutions then it follows that any combination

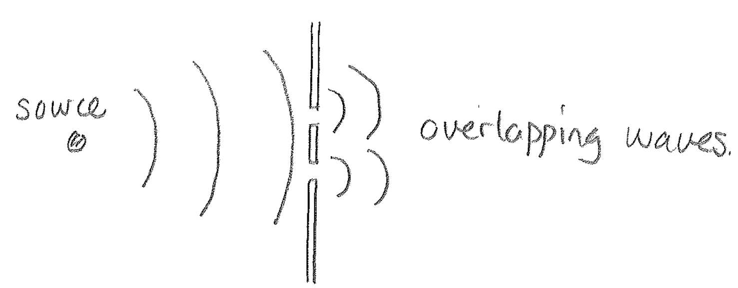
$$\vec{E}(\vec{r},t) = \alpha_1 \vec{E}_1 + \alpha_2 \vec{E}_2$$

where α_1 and α_2 are constants is also a solution. Such situations can easily be created when there are two or more sources of waves. Examples are:

1) two light sources/transmitters ...



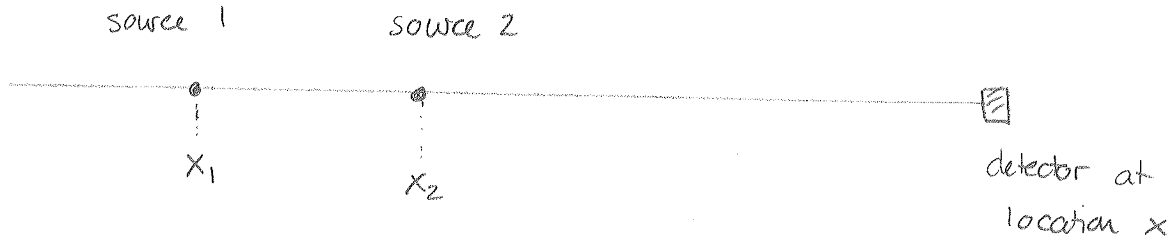
2) waves from a single source are split and combined as in the double slit experiment.



We can assess these in two closely related ways:

1) waves overlapping in space
2) waves overlapping in time at the detector.

The simplest situation to consider involves waves propagating along one direction and produced by two sources along the line of propagation.



We consider plane waves produced by the sources. These will have arbitrary frequencies, phases and amplitudes. However for each the wavevector is along $\hat{x}$. So $\vec{k}_1 = k_1 \hat{x}$ etc,... Then the waves are:

$$\vec{E}_1 = \vec{E}_{01} \, e^{i(k_1(x-x_1) - \omega_1 t + \phi_1)}$$

$$\vec{E}_2 = \vec{E}_{02} \, e^{i(k_2(x-x_2) - \omega_2 t + \phi_2)}$$

Exercise: a) Determine an expression for the superposition and use it to give an expression for the time dependent irradiance.

b) Determine an expression for the time-independent irradiance for the cases:

i) $\omega_1 \neq \omega_2$

ii) $\omega_1 = \omega_2$.

Answer: a) $\vec{E} = \vec{E}_1 + \vec{E}_2$

$$= \vec{E}_{10}\, e^{i(k_1(x-x_1)-\omega_1 t+\phi_1)} + \vec{E}_{20}\, e^{i(k_2(x-x_2)-\omega_2 t+\phi_2)}$$

Now the irradiance is determined via

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

and it is easily shown that

$$\vec{B} = \frac{1}{\omega_1} \vec{k}_1 \times \vec{E}_1 + \frac{1}{\omega_2} \vec{k}_2 \times \vec{E}_2$$

and this eventually gives

$$S = \epsilon v E^2$$

Using real parts of $\vec{E}$ gives:

$$S = \epsilon v \left(\vec{E}_{10} \cos\left[k_1(x-x_1)-\omega_1 t+\phi_1\right] + \vec{E}_{20} \cos\left[k_2(x-x_2)-\omega_2 t+\phi_2\right]\right)$$

$$\cdot\left(\ldots \text{ same } \ldots \right)$$

$$= \epsilon v \Big\{ E_{10}^2 \cos^2\left[k_1(x-x_1)-\omega_1 t+\phi_1\right]$$

$$2\, \vec{E}_{10}\cdot\vec{E}_{20} \cos\left[k_1(x-x_1)-\omega_1 t+\phi_1\right] \cos\left[k_2(x-x_2)-\omega_2 t+\phi_2\right]$$

$$+ E_{20}^2 \cos^2\left[k_2(x-x_2)-\omega_2 t+\phi_2\right]\Big\}$$

This clearly fluctuates with time.

b) We know that

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \cos^2[\omega t + \alpha]\, dt = \frac{1}{2}$$

We also need to calculate

$$\frac{1}{T} \int_{t_0}^{t_0+T} \cos[\omega_1 t + \alpha_1] \cos[\omega_2 t + \alpha_2]\, dt$$

Here $\cos(A)\cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$ gives:

$$\frac{1}{T}\int_{t_0}^{t_0+T} \cos[\dots] \cos[\dots] dt = \frac{1}{2}\frac{1}{T}\int_{t_0}^{t_0+T} \cos[(\omega_1+\omega_2)t + \alpha_1+\alpha_2]\, dt$$

$$+ \frac{1}{2}\frac{1}{T}\int_{t_0}^{t_0+T} \cos[(\omega_1-\omega_2)t + \alpha_1-\alpha_2]\, dt.$$

The first will give an expression of the form:

$$\frac{1}{T}\frac{1}{(\omega_1+\omega_2)} \sin[(\omega_1+\omega_2)t + \alpha_1+\alpha_2] \Big|_{t_0}^{t_0+T}$$

and if $T(\omega_1+\omega_2) \gg 1$ then this is approximately zero. The second depends on whether $\omega_1 = \omega_2$ or $\omega_1 \neq \omega_2$. If $\omega_1 \neq \omega_2$ then we get

$$\frac{1}{T}\frac{1}{\omega_1-\omega_2} \sin[(\omega_1-\omega_2)t + \alpha_1+\alpha_2] \Big|_{t_0}^{t_0+T} \to 0 \quad \text{as } T\to\infty$$

But if $\omega_1 = \omega_2$ then this will give

$$\frac{1}{2}\cos[\alpha_1+\alpha_2]$$

Thus we get:

If $\vec{E}_1 = \vec{E}_{10}\, e^{i(k(x-x_1)-\omega_1 t+\phi_1)}$

$\vec{E}_2 = \vec{E}_{20}\, e^{i(k(x-x_2)-\omega_2 t+\phi_2)}$

then if $\omega_1 \neq \omega_2$

$$I = \frac{1}{2}\epsilon v E_{10}^2 + \frac{1}{2}\epsilon v E_{20}^2$$

and if $\omega_1 = \omega_2$ (and $k_1 = k_2 \equiv k$)

$$I = \frac{1}{2}\epsilon v E_{10}^2 + \frac{1}{2}\epsilon v E_{20}^2 + \epsilon v\, \vec{E}_{10}\cdot\vec{E}_{20} \cos\left[k(x_2-x_1)+\phi_1-\phi_2\right]$$

This states that

If the waves have different frequencies then the intensity of the superposition is the sum of the intensities of the two individual waves

The more interesting case occurs when the waves have the same frequency. Then defining the phase difference between them as

$$\delta := k(x-x_2) - k(x-x_1) + \phi_2 - \phi_2 = k(x_1-x_2) + \phi_2 - \phi_1$$

we see that the intensity depends on the polarization and phase shift between the waves.

Thus

> If the sources produce waves of the same frequency then
>
> $$I = I_1 + I_2 + \epsilon v\, \vec{E}_{01} \cdot \vec{E}_{02} \cos \delta$$
>
> where
>
> I_1 = intensity of 1 alone
>
> I_2 = " " 2 alone.

The intensity clearly depends on the polarization. We consider two extremes of linear polarization:

1) if the polarization axes are orthogonal then $\vec{E}_{01} \cdot \vec{E}_{02} = 0$

$$\Rightarrow \quad I = I_1 + I_2$$

2) if the polarization axes are parallel then $\vec{E}_{01} \cdot \vec{E}_{02} = E_{01} E_{02}$

But $\frac{1}{2}\epsilon v E_{01}^2 = I_1 \Rightarrow \quad E_{01} = \sqrt{\frac{2I_1}{\epsilon v}}$

$\frac{1}{2}\epsilon v E_{02}^2 = I_2 \Rightarrow \quad E_{02} = \sqrt{\frac{2I_2}{\epsilon v}}$

So

If the sources produce linearly polarized waves whose polarization axes are parallel then

$$\boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \delta}$$