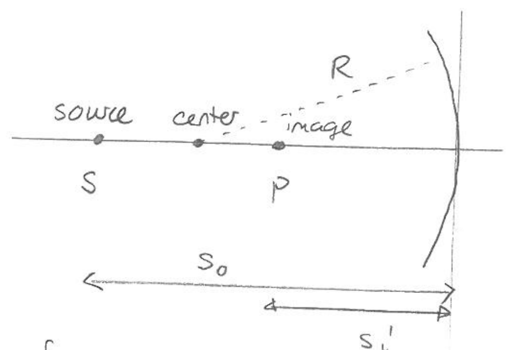
Phys 473 2015

Lecture 18

Reflection off spherical surfaces.

We saw that perfect reflection off a spherical surface can only occur for the trivial case where the source and image are at the center of the circle. However, there is a realm in which near perfect image formation occurs and this is when the angles involved are all small.
This is the paraxial approximation.
We found that the distance from mirror to image is given via:



$$\frac{1}{S} + \frac{1}{P} = \frac{2}{R}$$

We adopt the more conventional notation for a concave mirror:

S_o = distance from mirror surface to source (object)
S_i = " " " " " image.

Then for a concave mirror

$$\boxed{\frac{1}{S_o} + \frac{1}{S_i} = \frac{1}{f}}$$

where the focal length is

$$\boxed{f = R/2}$$

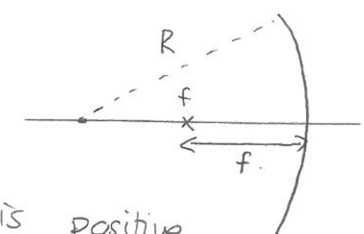
We can interpret the focal length by considering an extreme case: a source infinitely far from the mirror. So $s_o \to \infty$ and

$$\frac{1}{s_i} = \frac{1}{f} \Rightarrow s_i = f$$

This gives rise to:

$\boxed{\text{The focal point of a concave mirror is the point through which parallel rays from an infinitely distant object will pass after reflection.}}$

The focal length is the distance from the mirror to the focal point. This will be crucial in image formation by such mirrors.



Exercise: a) Suppose $s_o > f$. Describe whether s_i is positive or negative and if it increases or decreases as s_o increases.

b) Suppose $s_o < f$. Is s_i positive or negative?

c) Describe when $s_o = s_i$ in terms of the mirrors radius.

Answer: a) $\frac{1}{s_i} = \frac{1}{f} - \frac{1}{s_o} = \frac{s_o - f}{s_o f} \Rightarrow s_i = \frac{f s_o}{s_o - f} > 0$ if $s_o > f$

So s_i is positive

As s_o increases $\frac{1}{s_o}$ decreases $\Rightarrow -\frac{1}{s_o}$ increases $\Rightarrow \frac{1}{s_i}$ increases $\Rightarrow s_i$ decreases.

b) Here $s_i < 0$ by part a).

c) $\frac{1}{s_o} + \frac{1}{s_o} = \frac{1}{f} \Rightarrow \frac{2}{s_o} = \frac{1}{f} \Rightarrow s_o = 2f = R$ (center of the sphere).

Ray tracing for spherical concave mirrors

We are forced to interpret $s_i < 0$ when $s_o < f$. This and other mirror image formation can be understood by tracing rays. The ray tracing involves two rays:

i) a ray parallel to the optical axis – must pass through the focal point

ii) a ray passing through the focal point - must emerge parallel.

iii) a ray passing to the point where the mirror meets to optical axis – must be reflected at an equal angle.

In order that these diagrams respect the small angle approximation, the working surface of the mirror is taken to be a vertical line perpendicular to the optical axis – an exaggerated version of the actual surface is often drawn in.

Exercise: A mirror has focal length 25cm Use a meter stick to trace these rays for:

a) object located at 40cm

b) " " at 10cm

c) Use the equation to predict the image location. Do they agree?

Answer: a)

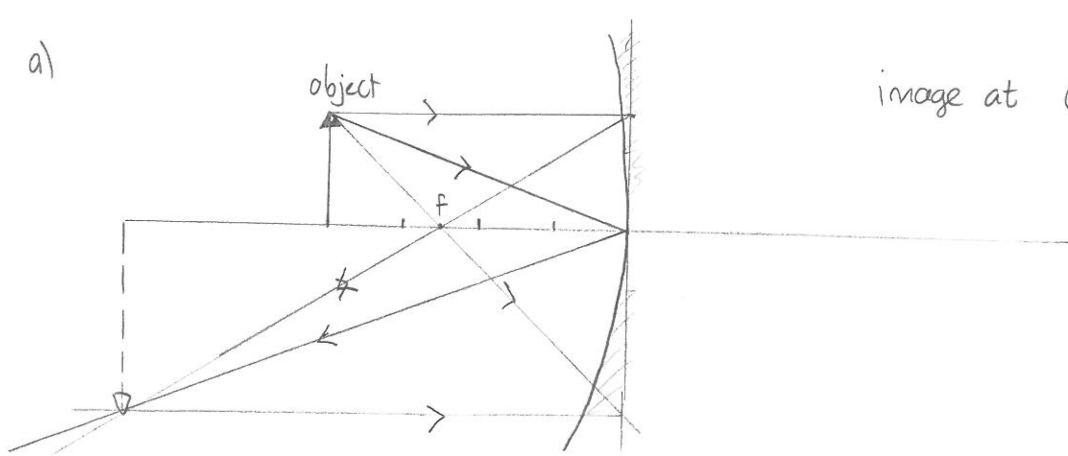
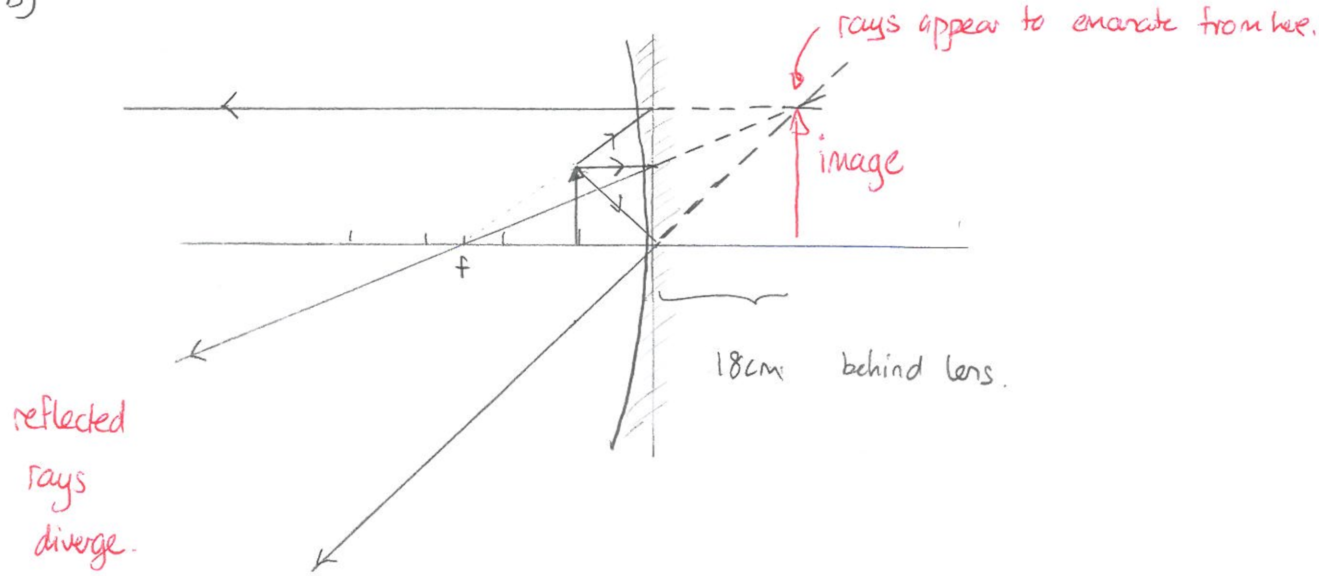


image at 66cm

b)



c) $S_o = 40\text{cm}$ $\qquad \frac{1}{40\text{cm}} + \frac{1}{S_i} = \frac{1}{25\text{cm}} \Rightarrow \frac{1}{S_i} = \frac{1}{25\text{cm}} - \frac{1}{40\text{cm}}$

$$\Rightarrow S_i = \frac{25\times 40}{40-25} = \frac{\overset{5}{\cancel{25}}\times 40}{\cancel{15}\,3} = 67\text{cm}.$$

$S_o = 10\text{cm}$ $\qquad \frac{1}{10\text{cm}} + \frac{1}{S_i} = \frac{1}{25\text{cm}} \Rightarrow \frac{1}{S_i} = \frac{1}{25\text{cm}} - \frac{1}{10\text{cm}}$

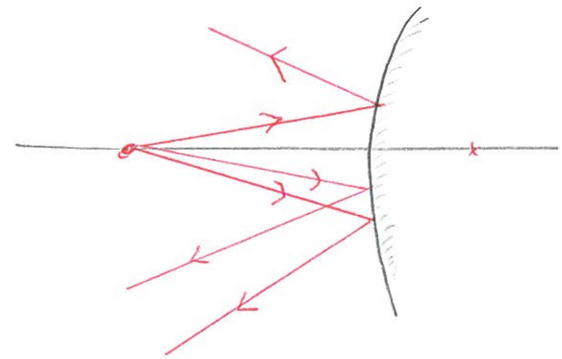
$$\Rightarrow S_i = \frac{25\times 10}{10-25} = -\frac{250}{15} = -17\text{cm}.$$

Thus for a concave mirror:

If object is beyond f, inverted real image is formed in front of mirror and $S_i > 0$ If object is between f + mirror, upright virtual image is formed behind mirror and $S_i < 0$.

Demo: Use bboard optics mirror – find when ~~vir~~ virtual image is no longer produced
– get f → get R → trace circle with string.

Convex mirrors.

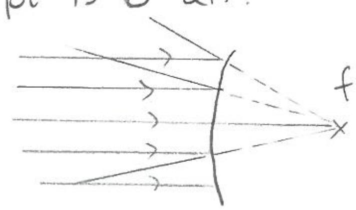


A convex mirror will clearly cause light rays to diverge after reflection and can never form a real image on the same side of the mirror as the source or object. However, the rays could appear to emanate from a point behind the mirror. For small angles the same analysis yields the same results as before:

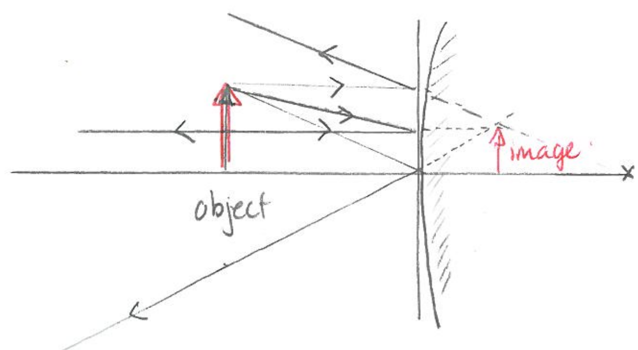
$$\frac{1}{S_o} + \frac{1}{S_i} = \frac{1}{f}$$

However, here the <u>focal length is negative</u> : focal pt is to left.

$f = -R/2$ convex



and $S_i < 0$. The same rays can be traced



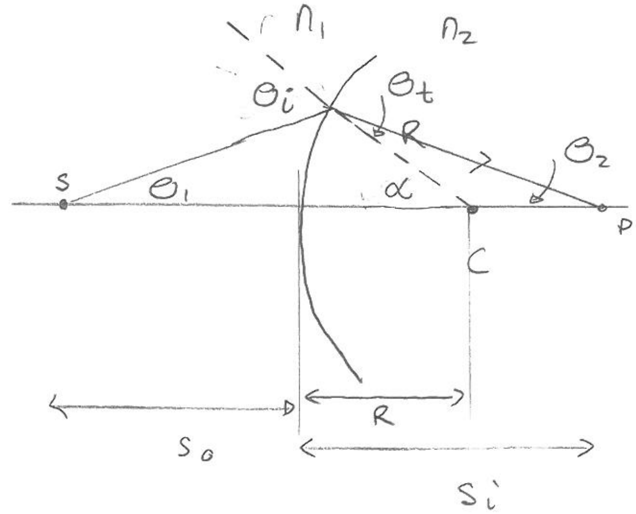
Then

A convex mirror always produces a virtual upright image behind the mirror. The image is always smaller than the object. $S_i < 0$.

Refraction by spherical surfaces.

Consider a source beyond a single spherical surface centered at C

Light from a source S is refracted to point P. To find this:



① θ_1 and S_o determine θ_i

② Snell's Law determines θ_t

③ θ_i, θ_1 determine θ_2

④ θ_2, R, θ_t determine S_i

To see this

① Law of sines with SQC

$$\Rightarrow \frac{\sin(\pi-\theta_i)}{S_o+R} = \frac{\sin\theta_1}{R}$$

$$\Rightarrow \frac{\sin\theta_i}{S_o+R} = \frac{\sin\theta_1}{R} \Rightarrow \sin\theta_i = \frac{S_o+R}{R}\sin\theta_1 \qquad -(A)$$

② $n_1 \sin\theta_i = n_2 \sin\theta_t \Rightarrow n_1 \dfrac{S_o+R}{R}\sin\theta_1 = n_2 \sin\theta_t$

$$\Rightarrow \sin\theta_t = \frac{n_1}{n_2}\frac{S_o+R}{R}\sin\theta_1$$

③ $\theta_2+\theta_t = \alpha$ and $\alpha+\theta_1 = \theta_i \Rightarrow \theta_2 = \theta_i-\theta_1-\theta_t$

④ Law of sines with PCQ $\Rightarrow \dfrac{\sin\theta_2}{R} = \dfrac{\sin\theta_t}{S_i-R} \Rightarrow \dfrac{S_i-R}{R} = \dfrac{\sin\theta_t}{\sin\theta_2}$

$$\Rightarrow S_i = R + R\frac{\sin\theta_t}{\sin\theta_2}$$

$$\Rightarrow S_i = R + \frac{n_1}{n_2}(S_o+R)\frac{\sin\theta_1}{\sin(\theta_i-\theta_1-\theta_t)}$$

These series of equations give a scheme for finding s_i exactly. We now consider where all angles are small, i.e. $\sin\theta \approx \theta$.

Also let

$$\boxed{n = \frac{n_2}{n_1}}$$

Then:

$$s_i \approx R + \frac{1}{n}(s_o + R)\frac{\theta_1}{\theta_i - \theta_t - \theta_1}$$

But (A) gives $\theta_i \approx \left(\frac{s_o}{R} + 1\right)\theta_1$

(B) gives $\theta_t \approx \frac{1}{n}\left(\frac{s_o}{R} + 1\right)\theta_1$

$$\Rightarrow \theta_i - \theta_t - \theta_1 = \left[\frac{s_o}{R} + \cancel{1} - \frac{1}{n}\left(\frac{s_o}{R} + 1\right) - \cancel{1}\right]\theta_1$$

and these yield

$$s_i \approx R + \frac{1}{n}(s_o + R)\frac{1}{\left[\frac{s_o}{R} - \frac{1}{n}\left(\frac{s_o}{R} + 1\right)\right]}$$

$$\Rightarrow s_i \approx R + \frac{1}{n}(s_o + R)\frac{R}{s_o - \frac{1}{n}(s_o + R)}$$

$$\left(s_i - R\right)\left(s_o - \frac{1}{n}(s_o + R)\right) = \frac{1}{n}(s_o + R)R$$

$$\Rightarrow s_i s_o - R s_o - \frac{1}{n} s_i (s_o + R) = 0$$

$$\Rightarrow s_i s_o \left(1 - \frac{1}{n}\right) - R s_o - \frac{1}{n} s_i R = 0$$

$$\Rightarrow s_i s_o (n-1) = s_i R + n R s_o$$

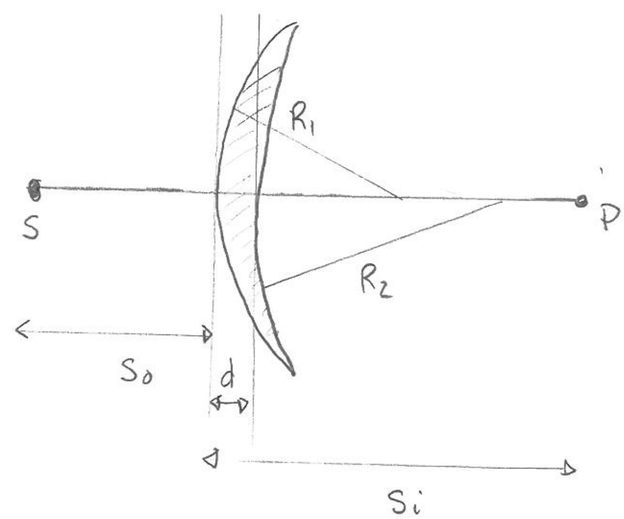
$$\Rightarrow \frac{(n-1)}{R} = \frac{1}{s_o} + \frac{n}{s_i}$$

We see that this is independent of θ_1 and so

> For small angles the spherical surface produces an approximately perfect image at s_i given via
>
> $$\frac{n}{s_i} + \frac{1}{s_o} = \frac{n-1}{R}$$
>
> where $n = n_2/n_1$

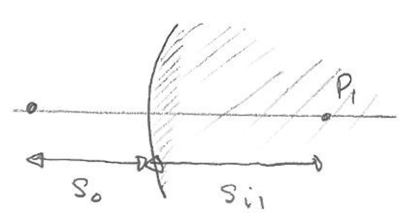
Usually with lenses, there are two surfaces with different radii. We can combine the previous result to show that with the illustrated set up:

> For a thin lens $(d \to 0)$
>
> $$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$$
>
> where the focal length satisfies
>
> $$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$
>
> where $n = n_{lens}/n_{medium}$ and n_{medium} is the index of refraction of the medium and n_{lens} of the lens.

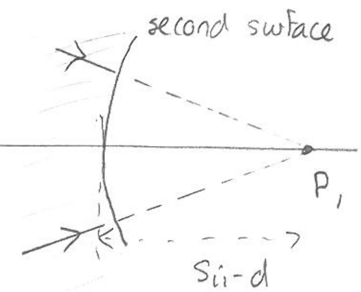


<u>Proof:</u> The first spherical surface (radius R_1) produces an image at point P_1 where

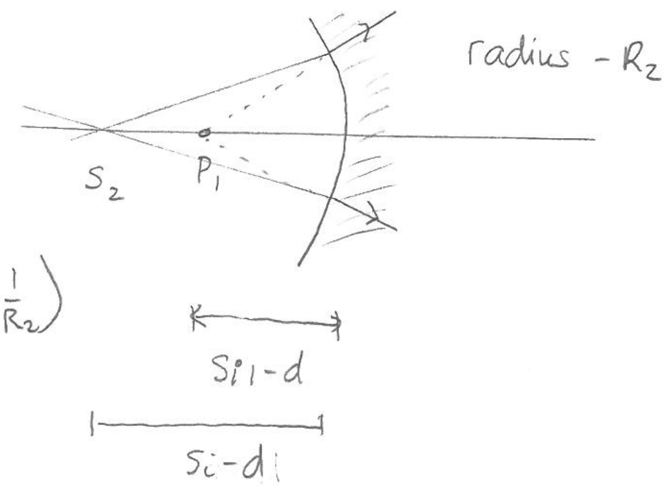
$$\frac{1}{s_o} + \frac{n}{s_{i1}} = \frac{n-1}{R_1}$$



This would be true if the medium of the lens extended back. This serves as a virtual source for the second surface. We could reverse the analysis to say that there is another source S_2 that would produce the rays that appear to pass through P_1. Then



$$\frac{n}{-(S_{i1}-d)} + \frac{1}{(S_i-d)} = \frac{n-1}{-R_2}$$



Adding these gives:

$$\frac{1}{S_o} + \frac{n}{S_{i1}} - \frac{n}{(S_{i1}-d)} + \frac{1}{(S_i-d)} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{S_o} + \frac{1}{S_i-d} + n\left(\frac{1}{S_{i1}} - \frac{1}{S_{i1}-d}\right) = (n-1)(\ldots)$$

$$\Rightarrow \frac{1}{S_o} + \frac{1}{S_i-d} + n \frac{d}{S_{i1}(S_{i1}-d)} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

If $d \ll S_i$, $d \ll S_{i1}$ then $S_i - d \to S_i$, $\frac{d}{S_{i1}} \to 0$ and

this gives the result.