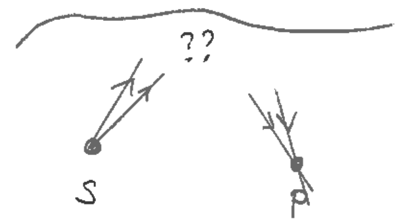
HW 12 due Weds

Phys 473 2015

Lecture 17

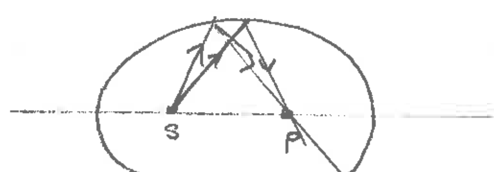
Reflection off aspherical surfaces

We sought situations where a reflecting surface produces a perfect image of a point source.



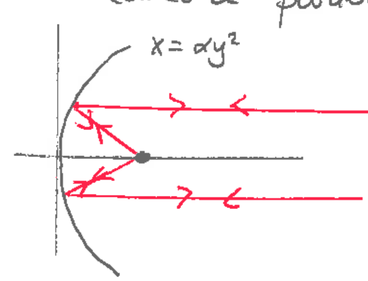
If the source is at location S and the detection point is P the requirement is that every ray emanating from S must pass through P. Then the issue is to determine a surface that produces this condition. We found:

The only surface that produces a perfect image of a point source is an ellipse. The source must be at a focus of the ellipse for this to occur. The image will be produced at the other focus.



For a given pair of focal points there are many possible ellipses varying in their semimajor + semiminor axes. One can see this by the condition that the total distance from one focus to any point on the curve and back to the other focus is constant. Varying this will generate ellipses of different sizes.

An extreme case is where $P \to \infty$. The ellipse becomes a parabola



Light rays emanating from the near focus travel parallel to the axis of the parabola

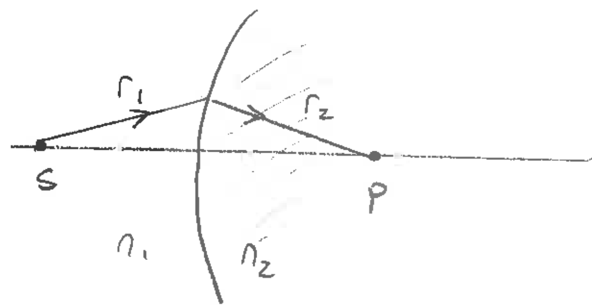
We have

$\boxed{\text{A parabolic mirror focusses light rays parallel to its axis to a point at the focal point of the parabola}}$

Refraction off aspherical surfaces

A similar analysis using Fermat's principle of least time can be brought to bear on refraction.

If the source and detection points are S and P. Then all rays traveling from s to p must take the same total time



$$t = \frac{r_1}{v_1} + \frac{r_2}{v_2}$$

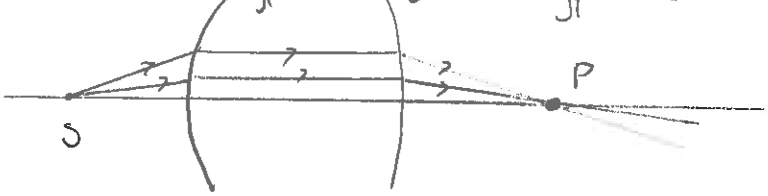
where v_i is the speed in medium i. But $v_i = c/n_i$ implies that

$$t = (n_1 r_1 + n_2 r_2)/c$$

and so the curve must be constructed such that $n_1 r_1 + n_2 r_2 = \text{const}$. This can be done and in the extreme as $P \to \infty$ it yields a curve of a hyperbola. But this is of less pratical interest that a reflecting surface since the optical element would need to be infinitely deep for this to work.

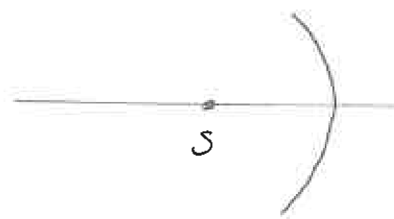
Nevertheless, this does also yield a perfect refracting surface for a pair of focal points. Where is could be useful is for a double surface

hyperbola different hyperbola

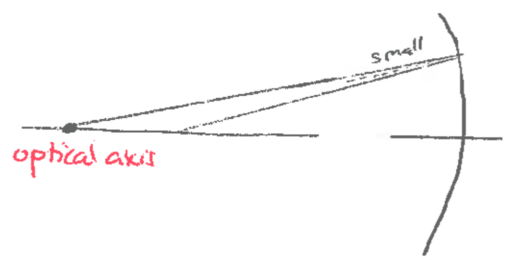


Reflection off a spherical surface

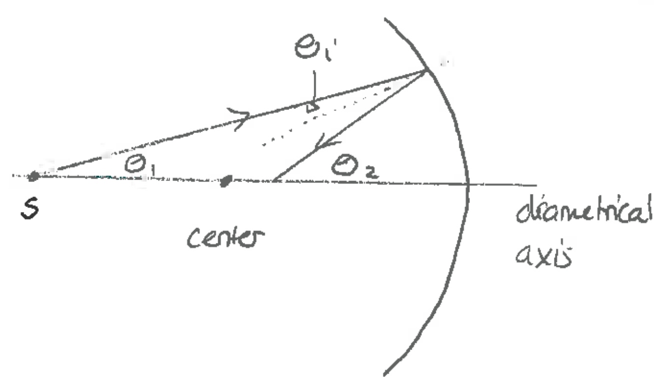
Many optical elements are not constructed from general elliptical or parabolic surfaces but rather from spherical surfaces. A sphere is an example of an ellipse with both foci at the same location: the center of the sphere. Thus there is only one source point which produces a perfect image — the center. And the image is produced at the center.



We will want to broaden this to find where approximate images are formed when the source is not at the center of the sphere. This will in general work when the spatial extent of the mirror is such that rays from the source make small angles of incidence at the reflecting surface.

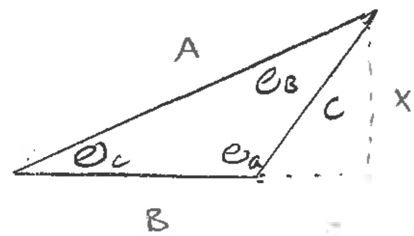


Consider, for the moment, a general case of a spherical surface of radius R. We consider a source S and ask: Where does the law of reflection predict that the ray will pass after reflection to the optical axis, which is a diametrical line through the center



The analysis is aided by a trigonometric rule called the law of sines. This states that

$$\frac{\sin\theta_a}{A} = \frac{\sin\theta_b}{B} = \frac{\sin\theta_c}{C}$$



Proof In the diagram

$$\frac{x}{A} = \sin\theta_c \qquad \frac{x}{C} = \sin(\pi - \theta_a)$$

$$= \cancel{\sin\pi\cos\theta_a} - \cos\pi\,\sin\theta_a \quad = \sin\theta_a$$

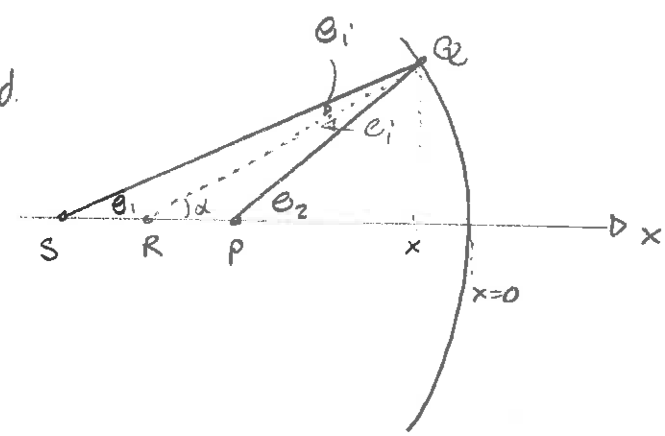
$$\Rightarrow \quad \begin{array}{l} x = A\sin\theta_c \\ x = C\sin\theta_a \end{array} \quad \Rightarrow \quad A\sin\theta_c = C\sin\theta_a$$

$$\Rightarrow \quad \frac{\sin\theta_a}{A} = \frac{\sin\theta_c}{C}$$

with similar results for θ_B and B $\boxtimes$

Now consider the situation in more detail.
We label locations along the axis with their co-ordinates as illustrated.

Then geometrical rules will relate θ_1, θ_2 and θ_i while trigonometric rules will relate s, p, x and the angles



The scheme is

1. S, R, θ_1 determine θ_i
2. θ_1, θ_i, S, R determine x
3. θ_1, θ_i determine θ_2
4. θ_1, θ_2, S, x determine P

Exercise. Suppose the radius of the mirror is 1 and $S = 1.2$ from mirror. Consider the ray where $\theta_1 = 5.00°$

a) Determine θ_i

b) Determine x

c) Determine θ_2

d) Determine P

Answer: a) Using SQR

$$\frac{\sin\theta_i}{S-R} = \frac{\sin\theta_1}{R} \Rightarrow \sin\theta_i = \frac{S-R}{R}\sin\theta_1$$

$$= \frac{0.2}{1.0}\sin 5.0°$$

$$\Rightarrow \theta_i = \sin^{-1}(0.0174)$$

$$= 1.00°$$

b) Using QSx and h for height of Q.

$$\left.\begin{array}{l}\dfrac{h}{S-x} = \tan\theta_1 \\[2ex] \dfrac{h}{R-x} = \tan\alpha\end{array}\right\} \Rightarrow (S-x)\tan\theta_1 = (R-x)\tan\alpha$$

But $\alpha = \theta_1 + \theta_i$

$$\Rightarrow (S-x)\tan\theta_1 = (R-x)\tan(\theta_1+\theta_i)$$

$$\Rightarrow x\left[\tan(\theta_1+\theta_i) - \tan\theta_1\right]$$

$$= R\tan(\theta_1+\theta_i) - S\tan\theta_1$$

$$\Rightarrow x = \frac{R\tan(\theta_1+\theta_i) - S\tan\theta_1}{\tan(\theta_1+\theta_i) - \tan\theta_1}$$

$$= \frac{1.00\tan(6.00°) - 1.2\tan(5.00°)}{\tan(6.00°) - \tan(5.00°)} = \frac{0.000118}{0.0176}$$

$$= 0.00670$$

c) $\theta_2 = \theta_1 + 2\theta_i = 7.00°$

d) $\dfrac{h}{P-x} = \tan\theta_2 \Rightarrow h = (P-x)\tan\theta_2$

$\dfrac{h}{S-x} = \tan\theta_1 \Rightarrow h = (S-x)\tan\theta_1$

$$\Rightarrow (P-x)\tan\theta_2 = (S-x)\tan\theta_1$$

$$\Rightarrow P-x = (S-x)\frac{\tan\theta_1}{\tan\theta_2} \Rightarrow P = x + (S-x)\frac{\tan\theta_1}{\tan\theta_2}$$

Here $\qquad p = 0.00670 + (1.2 - 0.00670)\dfrac{\tan(5.00°)}{\tan(7.00°)}$

$$= 0.857 \qquad \boxed{2}$$

This illustrates the workings of the scheme. We can see that they chain together via

$$p = x + (s-x)\frac{\tan\theta_1}{\tan(\theta_1 + 2\theta_i)}$$

$$x = \frac{R\tan(\theta_1+\theta_i) - s\tan\theta_1}{\tan(\theta_1+\theta_i) - \tan\theta_1}$$

$$\sin\theta_i = \frac{s-R}{R}\sin\theta_1$$

This is clearly convoluted in general. We aim to consider small angles θ_1 and θ_i (in radians). In this case

$$\left.\begin{aligned} \sin\theta &\approx \theta \\ \tan\theta &\approx \theta \end{aligned}\right\} \text{ paraxial approx}$$

<u>Exercise:</u> Use the paraxial approximation to determine simple expressions for:

a) θ_i vs θ_1

b) x

c) p

Answer: a) $\sin\theta_i = \frac{s-R}{R}\sin\theta_1$

$$\Rightarrow \theta_i \approx \frac{s-R}{R}\theta_1$$

$$\Rightarrow \theta_i \approx \frac{s}{R}\theta_1 - \theta_1 \quad \Rightarrow \quad \theta_1 + \theta_i = \frac{s}{R}\theta_1$$

b) $$x = \frac{R\tan(\theta_1+\theta_i) - s\tan\theta_1}{\tan(\theta_1+\theta_i) - \tan\theta_1}$$

$$\approx \frac{R(\theta_1+\theta_i) - s\theta_1}{\cancel{\theta_1} + \theta_i - \cancel{\theta_1}} = \frac{R\frac{s}{R}\theta_1 - s\theta_1}{\theta_i} = 0$$

c) $$p = x + (s-x)\frac{\tan\theta_1}{\tan(\theta_1+2\theta_i)}$$

$$\approx \frac{s\theta_1}{\theta_1+2\theta_i} = \frac{s\theta_1}{\theta_1 + 2\left(\frac{s}{R}\theta_1 - \theta_1\right)}$$

$$\Rightarrow p = \frac{s\theta_1}{2\frac{s}{R}\theta_1 - \theta_1} = \frac{sR}{2s-R} \Rightarrow p = \frac{1}{2/R - 1/s}$$

$$\Rightarrow \frac{1}{p} = \frac{2}{R} - \frac{1}{s}$$

The last gives

For small angles all rays (approximately) pass through the same point and this satisfies:

$$\boxed{\frac{1}{p} = \frac{2}{R} - \frac{1}{s}}$$

We will rephrase this in more conventional terms next time.