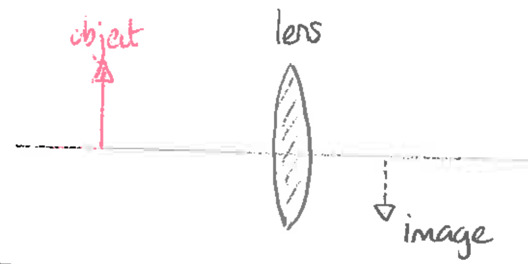
HW due Friday

Geometric Optics: Optical Elements

Optical elements such as lenses and mirrors are devices that manipulate light in a co-ordinated fashion. We consider the operation of elements that use reflection or refraction to do this (as opposed to polarization or non-linear effects). Our aim is to analyze their behavior using basic rules of reflection + refraction.

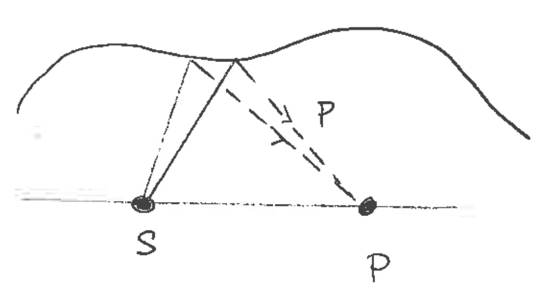
For example a lens produces images of objects using refraction. The way in which this unfolds partly depends on the location of the object relative to the lens but it also depends on the shape of the surface of the lens.



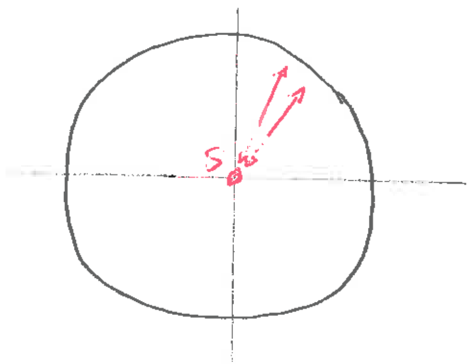
We will mostly focus on optical elements whose surfaces are spherical, or sections of spheres. However, we begin by considering aspherical surfaces

Reflection at aspherical surfaces

Consider light rays traveling in a plane. Suppose they are emitted from one source S and travel to a surface. We are interested in the situation where these are all reflected back to a common detection point P



Clearly one surface that allows this is a sphere, but only provided that the source is at the center of the sphere. Then each ray strikes



the surface with angle $\theta_i = 0°$ from the normal. It is reflected back at angle $\theta_r = 0°$ from the normal. These rays converge at the center. A geometrical construction would reveal that if the source were anywhere but the center, then not all rays pass through the same point.

The question now is what other surfaces allow this.

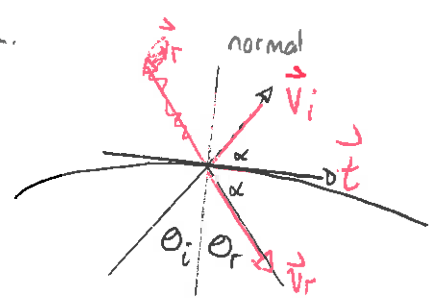
In general there are two considerations:

i) according to Fermat's principle the time traveled by all paths from S to P must be minimal and it must be the same.

ii) the law of reflection must apply at the surface.

These two rules actually yield the same information.

<u>Proof:</u> Consider reflection at the surface.



We require the same angles θ

Let $\vec{t}$ be the vector tangent to the surface. Then

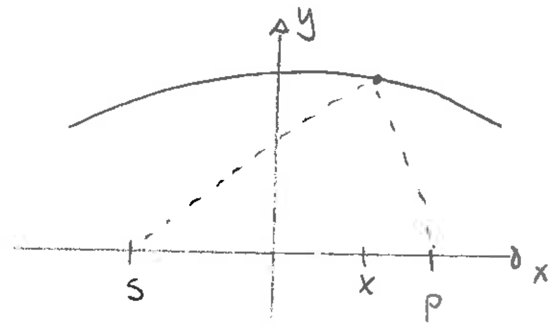
$$\vec{t} = \hat{x} + \frac{dy}{dx}\hat{y}$$

is an unnormalized tangent vector

If $\theta_i = \theta_r$ then the two angles α are equal. So

$$\hat{v}_i \cdot \vec{t} = \hat{v}_r \cdot \vec{t}$$

Using the indicated co-ordinates.



$$\vec{v}_i = (x-s)\,\hat{x} + y\,\hat{y}$$

$$\Rightarrow \quad \hat{v}_i = \frac{(x-s)\,\hat{x} + y\,\hat{y}}{\sqrt{(x-s)^2 + y^2}}$$

and $$\hat{v}_r = \frac{-(x-p)\,\hat{x} - y\,\hat{y}}{\sqrt{(x-p)^2 + y^2}}$$

So $$\hat{v}_i \cdot \vec{t} = \vec{v}_r \cdot \vec{t} \iff \frac{(x-s) + y'y}{\sqrt{(x-s)^2 + y^2}} = \frac{-(x-p) - y'y}{\sqrt{(x-p)^2 + y^2}}$$

$$\iff \frac{d}{dx}\sqrt{(x-s)^2 + y^2} = -\frac{d}{dx}\sqrt{(x-p)^2 + y^2}$$

$$\iff \frac{d}{dx}\sqrt{\quad} + \sqrt{\quad} = 0$$

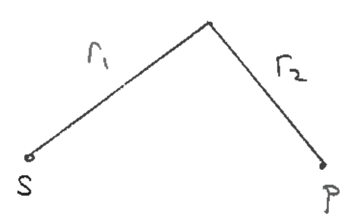
$$\iff \sqrt{(x-s)^2 + y^2} + \sqrt{(x-p)^2 + y^2} = \text{const}$$

The l.h.s is exactly the path length traveled $\boxtimes$

An alternative way to describe this is that the distance from the points

$r_1 + r_2$

is constant

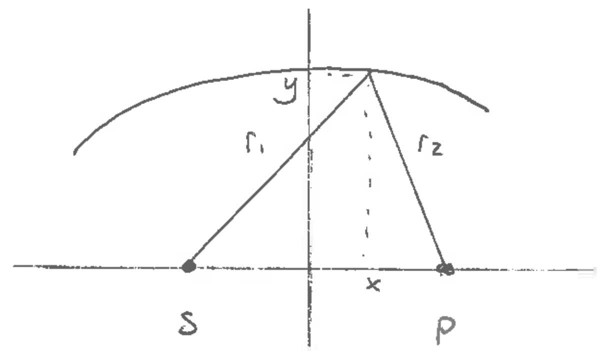


Demo: String with two fix ends and pencil at apex. Trace out curve

Image forming aspheric surfaces

The surface forms an image if either of the equivalent previous conditions is satisfied. We aim to find the equation that describes this surface. To simplify the calculation we arranged the co-ordinate axes so that the origin is midway between the source and detector points. Denote the co-ordinates of these points by

s and $-s$



Exercise: a) Find an expression for the total distance $r_1 + r_2$ in terms of x, y, s and a constant

b) Manipulate this to generate an expression.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a, b are constant for this curve.

Answer: a) $r_1 = \sqrt{(x-(-s))^2 + y^2}$

$r_2 = \sqrt{(x-s)^2 + y^2}$

$$r_1 + r_2 = \alpha = \text{const} \Rightarrow \sqrt{(x+s)^2 + y^2} + \sqrt{(x-s)^2 + y^2} = \alpha$$

b) Square both sides:

$$\left(\sqrt{(x+s)^2 + y^2} + \sqrt{(x-s)^2 + y^2}\right)^2 = \alpha^2$$

$$\Rightarrow (x+s)^2 + y^2 + (x-s)^2 + y^2 + 2\sqrt{\quad}\sqrt{\quad} = \alpha^2$$

$$\Rightarrow \quad x^2+s^2+2xs+y^2+x^2+s^2-2xs+y^2+2\sqrt{+}\sqrt{-}=\alpha^2$$

$$\Rightarrow \quad (x^2+s^2+y^2)+\sqrt{+}\sqrt{-}=\alpha^2/2$$

$$\Rightarrow \quad \sqrt{+}\sqrt{-}=\alpha^2/2-(x^2+y^2+s^2)$$

Square again

$$\left[(x+s)^2+y^2\right]\left[(x-s)^2+y^2\right]=\left[\frac{\alpha^2}{2}-(x^2+y^2+s^2)\right]^2$$

$$\Rightarrow \quad (x^2+2xs+s^2+y^2)(x^2-2xs+s^2+y^2)=\cdots$$

$$\Rightarrow \quad \cancel{(x^2+s^2+y^2)^2}-4x^2s^2=\frac{\alpha^4}{4}-\alpha^2(x^2+y^2+s^2)+\cancel{(x^2+y^2+s^2)^2}$$

$$\Rightarrow \quad x^2(\alpha^2-4s^2)+y^2\alpha^2=\alpha^2\left(\frac{\alpha^2}{4}-s^2\right)=\frac{1}{4}\alpha^2(\alpha^2-4s^2)$$

A geometrical argument (setting $y=0$) gives $\alpha>2s$. So we have

$$\frac{x^2}{\alpha^2/4}+\frac{y^2}{(\alpha^2-4s^2)/4}=1$$

This proves the result with $a=\alpha/2$

$b=\sqrt{\alpha^2-4s^2}/2$

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

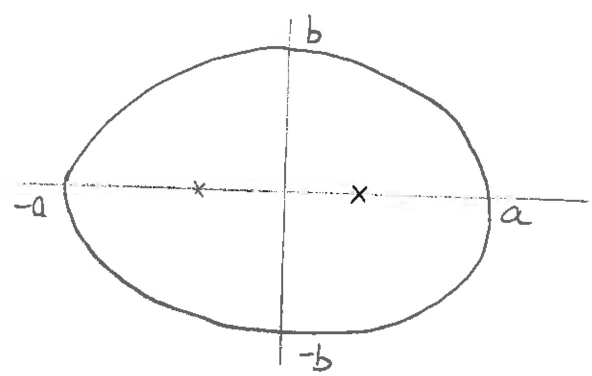
describes an ellipse whose center is at the origin. The points S and P are called the foci of the ellipse.

Exercise: a) Determine the extreme values of x and also y for the ellipse.

b) Given a, b determine an expression for the distance between the foci.

Answer: a) max x when $y = 0$ $\Rightarrow$ $x^2/a^2 = 1$ $\Rightarrow$ $x = \pm a$

max y when $x = 0$ $\Rightarrow$ $y^2/b^2 = 1$ $\Rightarrow$ $y = \pm b$.



b) The distance between the foci is $2s$. Using the rule $r_1 + r_2 = \text{const}$ we have for the point $x=0$ $y=b$

$$\sqrt{s^2+b^2} + \sqrt{s^2+b^2} = \text{const}$$

and for $x=a$ $y=b$

$$a - s + a + s = \text{const} \Rightarrow \text{const} = 2a.$$

So $\quad \cancel{2}\sqrt{s^2+b^2} = \cancel{2}a$

$$\Rightarrow \quad s^2+b^2 = a^2 \quad \Rightarrow \quad s^2 = a^2-b^2 \quad \Rightarrow \quad s = \sqrt{a^2-b^2}$$

(this assumes $a > b$) $\boxtimes$

It follows that a mirror with an elliptical surface will form a perfect image of any object placed at one focus point.

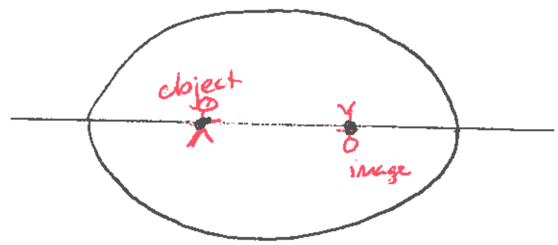
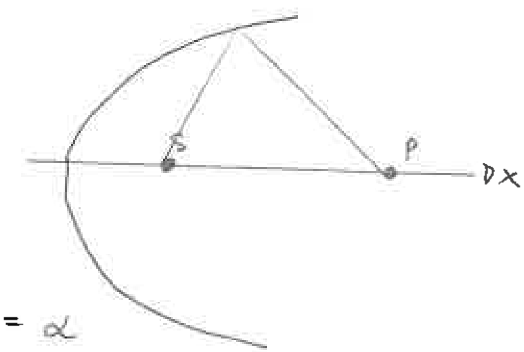


Image formation by a parabolic surface

A parabolic surface is an extreme example of an ellipse. We consider a general requirement for an ellipse with foci at

$0 < s < p < \infty$



Then $\quad r_1 + r_2 = \alpha = \text{const}$

$$\Rightarrow \quad \sqrt{(x-s)^2+y^2} + \sqrt{(x-p)^2+y^2} = \alpha$$

Again we would like an equation, but in the case where $p \to \infty$. To simplify the calculation we choose co-ordinates when $x=y=0$ is at the leftmost point of the curve.

Exercise a) Find an expression for α in terms of s, p.

b) Consider the limit as $p \to \infty$. Simplify the resulting expression

c) sketch light rays that originate from s and are reflected.

Answer: a) With $x = y = 0$ $\quad \sqrt{s^2} + \sqrt{p^2} = \alpha \Rightarrow \alpha = s + p$.

b) $\sqrt{(x-s)^2 + y^2} + \sqrt{(x-p)^2 + y^2} = s + p$

As $p \to \infty$ $\quad \sqrt{(x-s)^2 + y^2} + \sqrt{(p-x)^2} = s + p$

$$\Rightarrow \sqrt{(x-s)^2 + y^2} + \cancel{p} - x = s + \cancel{p}$$

$$\Rightarrow \sqrt{(x-s)^2 + y^2} = (x+s)$$

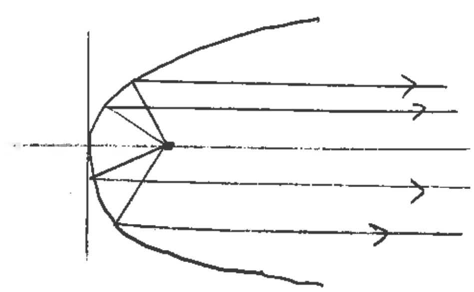
$$\Rightarrow (x-s)^2 + y^2 = (x+s)^2$$

$$\Rightarrow -2xs + y^2 = 2xs$$

$$\Rightarrow y^2 = 4xs \Rightarrow y = \sqrt{4xs}$$
$$= 2\sqrt{s}\sqrt{x}$$

This is an equation of a parabola $x = \frac{1}{4s} y^2$

c) The light rays are focussed to a point infinitely far away.



These parabolic mirrors form an image of an object that is infinitely far from the mirror. The object must be on the axis of the mirror and the image will be formed at the focus. An off axis object will result in an image formed elsewhere. And there will be some distortion in the image