Phys 473 2015

Lecture 15

Fri: HW due

Mon: Exam Covers Ch1-3

Emphasis on Ch 2-3

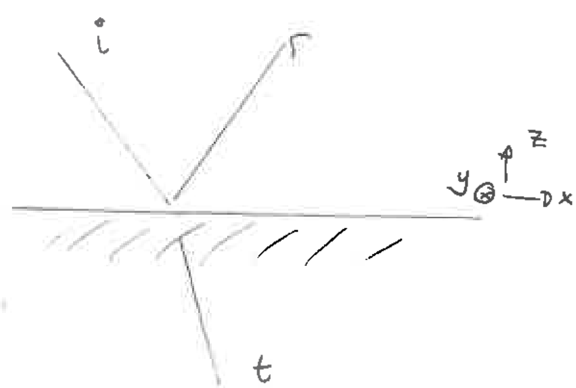
Review HW problems

Bring - 1/2 letter sheet single side - will supply inside covers of Griffiths

- calculator

Incidence Beyond the Critical Angle

We have analyzed the Fresnel equations and amplitude ratios whenever the angle of incidence is less than the critical angle. We now consider the situation when $\theta_i > \theta_c$. To do this we return to Maxwell's boundary conditions at the interface. The three waves are described by:



$$\vec{E}_i = \vec{E}_{oi}\, e^{i(\vec{k}_i\cdot\vec{r} - \omega_i t)}$$

$$\vec{E}_r = \vec{E}_{or}\, e^{i(\vec{k}_r\cdot\vec{r} - \omega_r t)}$$

Again, each of these satisfies the wave equation provided that

$$\omega^2 = \vec{k}\cdot\vec{k}\, v^2 \qquad \text{(dispersion relation)}$$

where v is the speed of waves in each medium

In what follows we will generalize the notion of such waves by allowing the wave vectors to be complex. In some situations we will see that these can still satisfy the wave equation but they yield solutions with a very different physical nature.

<u>Example:</u> Suppose that

$$\vec{k} = k_x \hat{x} - i K \hat{z}$$

where k_x, K are real and positive

a) Determine a condition that these must satisfy so that the dispersion relation is still valid.

b) If $\vec{E}_{oi} = \vec{A} e^{i\varphi}$ where $\vec{A}$ is real, determine the real solution to the wave equation.

<u>Answer:</u> a) $\vec{k}\cdot\vec{k} = k_x^2 + (-iK)^2 = k_x^2 - K^2$

Since both ω^2 and v^2 are positive, we require

$$k_x^2 - K^2 > 0 \quad \Rightarrow \quad k_x > K$$

b)
$$\vec{E} = \vec{A} e^{i\varphi} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$= \vec{A} e^{i\varphi} e^{i(k_x x - iKz - \omega t)}$$

$$= \vec{A} e^{+Kz} e^{i(k_x x - \omega t + \varphi)}$$

Then the real solution is:

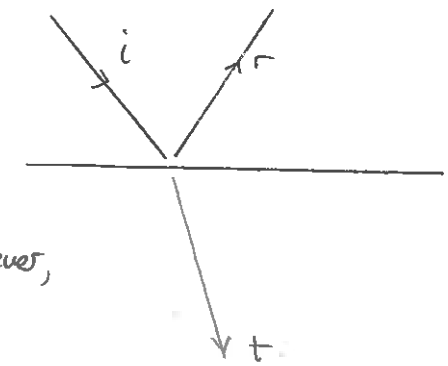
$$Re(\vec{E}) = Re\left[A e^{+Kz} \cos(k_x x - \omega t + \varphi) + i A e^{-Kz} \sin(k_x x - \omega t + \varphi)\right]$$

$$= A e^{+Kz} \cos(k_x x - \omega t + \varphi)$$

We see that this generalized solution can represent a wave which decays and thus corresponds to a decaying energy flux.

We now apply this to the issue of fields at the boundary between two media. The setup is as before and the boundary conditions for Maxwell's equations are as before. We assume that the incident wave is described by

$$\vec{E} = \vec{E}_{0i}\, e^{i(\vec{k}_i \cdot \vec{r} - \omega_i t)}$$



where $\vec{k}_i$ is real (and thus the wave does not demonstrate any decay). However, we could assume $\vec{E}_{0i}$ is complex.

Then we again check solutions of the form

$$\vec{E}_r = \vec{E}_{0r}\, e^{i(\vec{k}_r \cdot \vec{r} - \omega_r t)}$$

$$\vec{E}_t = \vec{E}_{0t}\, e^{i(\vec{k}_t \cdot \vec{r} - \omega_t t)}$$

where $\vec{k}_r$ and $\vec{k}_t$ could possible be complex. The dispersion relation yields:

$$\omega_i^2 = \vec{k}_i \cdot \vec{k}_i\, v_i^2$$

$$\omega_r^2 = \vec{k}_r \cdot \vec{k}_r\, v_r^2$$

$$\omega_t^2 = \vec{k}_t \cdot \vec{k}_t\, v_t^2$$

and here we note the crucial fact that $v_i = v_r$ since these two media are identical.

Applying the boundary conditions.

As before the boundary conditions result in expressions of the form:

$$\vec{E}_{0i}^{\,\parallel} e^{i(\vec{k}_i\cdot\vec{r}-\omega_i t)} + \vec{E}_{0r}^{\,\parallel} e^{i(\vec{k}_r\cdot\vec{r}-\omega_r t)} = \vec{E}_{0t}^{\,\parallel} e^{i(\vec{k}_t\cdot\vec{r}-\omega_t t)}$$

and this is only possible for all t and $r = x\hat{x} + y\hat{y}$ if

i) $\boxed{\omega_i = \omega_r = \omega_t = \omega}$

ii) $\boxed{\begin{aligned} k_{ix} &= k_{rx} = k_{tx} \\ k_{iy} &= k_{ry} = k_{ty} \end{aligned}}$

with no further restriction on the z components of the wave vectors. The immediate implication is that if the incident wave is an undecaying plane wave then the x and y components of all the wave vectors are real.

Another implication resulting from the dispersion relations is

$$\begin{aligned} \omega^2 &= k_i^2 v_i^2 \\ \omega^2 &= k_r^2 v_r^2 \end{aligned} \quad \Rightarrow \quad k_i^2 v_i^2 = k_r^2 v_r^2$$

But $v_i^2 = v_r^2$. Thus we get

$$\boxed{k_r^2 = k_i^2}$$

and also $k_i^2 v_i^2 = k_t^2 v_t^2$ with $v_i = c/n_i$ and $v_t = c/n_t$ gives:

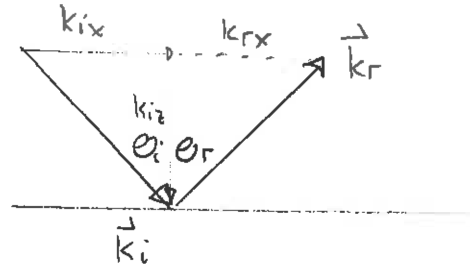
$$k_i^2 c^2/n_i^2 = k_t^2 c^2/n_t^2$$

$$\Rightarrow \boxed{k_t^2 = \left(\frac{n_t}{n_i}\right)^2 k_i^2}$$

We use these facts to ascertain as much as possible about $\vec{k_r}$ and $\vec{k_t}$.

Reflection

Here



$$k_r^2 = k_i^2$$

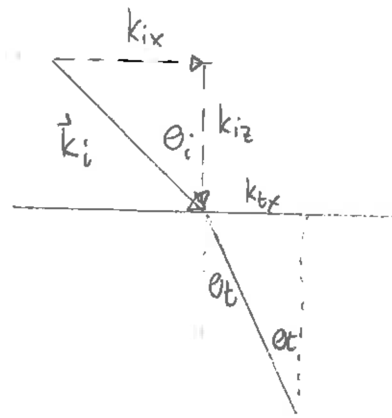
$$\Rightarrow k_{rx}^2 + k_{rz}^2 = k_{ix}^2 + k_{iz}^2$$

But $k_{rx} = k_{ix} \Rightarrow k_{rz}^2 = k_{iz}^2$ and it follows that $k_{rz} = \pm k_{iz}$. Only the negative sign refers to a reflected wave. This generates the usual law of reflection.

$$\theta_i = \theta_r$$

Transmitted light

Here



$$k_t^2 = \left(\frac{n_t}{n_i}\right)^2 k_r^2$$

$$\Rightarrow \left(k_{tx}^2 + k_{tz}^2\right) = \left(\frac{n_t}{n_i}\right)^2 \left(k_{ix}^2 + k_{iz}^2\right)$$

Exercise Rewrite this in terms of k_{ix}, $\sin\theta_i$ to get an expression for k_{tz}^2.

Describe when this is positive or not.

Answer: $$k_{tz}^2 = \left(\frac{n_t}{n_i}\right)^2 \left[k_{ix}^2 + k_{iz}^2\right] - \underbrace{k_{tx}^2}_{k_{ix}^2}$$

$$= k_{ix}^2 \left[\left(\frac{n_t}{n_i}\right)^2 - 1\right] + k_{iz}^2 \left(\frac{n_t}{n_i}\right)^2$$

But $\dfrac{k_{ix}}{k_{iz}} = \tan\theta_i$ gives

$$k_{tz}^2 = k_{ix}^2 \left\{\left(\frac{n_t}{n_i}\right)^2 - 1 + \frac{1}{\tan^2\theta_i}\left(\frac{n_t}{n_i}\right)^2\right\}.$$

$$= k_{ix}^2 \left[\left(\frac{n_t}{n_i}\right)^2 \left(\frac{\tan^2\theta_i + 1}{\tan^2\theta_i}\right) - 1\right]$$

$$= k_{ix}^2 \left[\left(\frac{n_t}{n_i}\right)^2 \frac{\sin^2\theta_i/\cos^2\theta_i + 1}{\sin^2\theta_i/\cos^2\theta_i} - 1\right]$$

$$= k_{ix}^2 \left[\left(\frac{n_t}{n_i}\right)^2 \frac{1}{\sin^2\theta_i} - 1\right]$$

Then this is positive $\iff \left(\dfrac{n_t}{n_i}\right)^2 > \sin^2\theta_i$

$$\iff \frac{n_t}{n_i} > \sin\theta_i \qquad \boxtimes$$

So we see that if $\dfrac{n_t}{n_i} < \sin\theta_i$ or $\theta_i > \theta_c$ then k_{tz}^2 is negative. This is the case that interests us. So

$$\boxed{\begin{array}{l}\text{If } \sin\theta_i > n_t/n_i \text{ then define} \\ K := k_{ix}\sqrt{1 - \frac{1}{\sin^2\theta_i}\left(\frac{n_t}{n_i}\right)^2}\end{array}}$$

Then $\quad k_{tz} = -iK$ and

$$\vec{k} = k_{ix}\hat{x} - iK\hat{z}$$

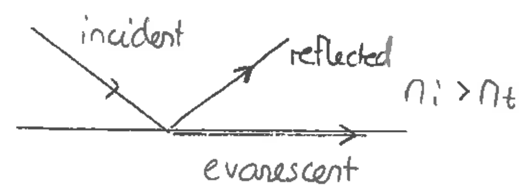
By the previous derivation it follows that the transmitted wave is described for $z < 0$ by either

$$\vec{E} = \vec{E}_{0i}\, e^{Kz}\, e^{i(k_{ix}x - \omega t)}$$

or if $\vec{E} = \vec{A}e^{i\epsilon}$ by

$$\boxed{\vec{E} = \vec{A}\, e^{Kz}\cos(k_{ix}x - \omega t + \epsilon)}$$

Thus there is actually a transmitted wave which travels parallel to the surface but whose intensity decays exponentially with surface depth. This is called an <u>evanescent wave</u>



We can modify K by noting that $\quad k_{ix} = k_i \sin\theta_i$ and so:

$$K = k_i \sin\theta_i \sqrt{1 - \frac{1}{\sin^2\theta_i}\left(\frac{n_t}{n_i}\right)^2} = \underbrace{k_i}_{\frac{n_i}{n_t}k_t} \sqrt{\sin^2\theta_i - \frac{n_t^2}{n_i^2}}$$

$$\Rightarrow \boxed{K = \frac{2\pi}{\lambda_i}\sqrt{\sin^2\theta_i - \frac{n_t^2}{n_i^2}}}$$

$$\Rightarrow K = k_t\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1} = \frac{2\pi}{\lambda_t}\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1}$$

and this determines the depth to which the wave penetrates. Note that

i) as λ_i increases, K decreases (so it's more pronounced at the blue end)

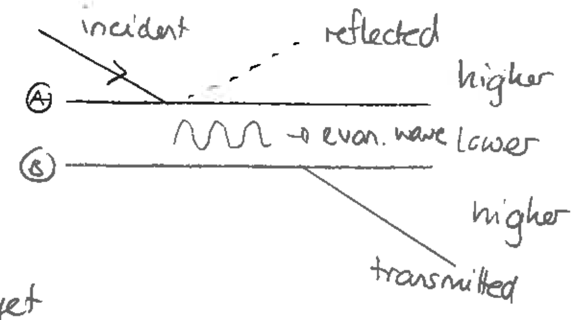
ii) as θ_i increases, K increases - much beyond the critical angle gives less penetration.

iii) as n_t/n_i decreases, K increases - greater disparity $\Rightarrow$ less penetration

Observing evanescent waves.

Evanescent waves can be observed by sandwiching a medium of lower index between two of higher index.

If the lower index medium is sufficiently narrow then light from the evanescent wave can reach the (B) boundary and then be transmitted through this barrier. Thus not all the light is reflected as one would usually get with total internal reflection. This is called frustrated total internal reflection (FTIR) and has been observed.



See Am J Phys 54, 601 (1986)

Am J Phys 71, 670 (2003)

Exercise Consider light of wavelength 632nm (in air). Suppose that this propagates from glass to air (glass index =1.50). Determine the depth at which the evanescent wave is $\frac{1}{e}$ at the surface as a function of θ_i. Calculate this for $\theta_i = 60°, 75°, 45°$.

Ans: Want $Kz = -1 \Rightarrow z = -1/K \Rightarrow 1/K$ = depth

$$1/K = \frac{\lambda_t}{2\pi} \Big/ \sqrt{\left(\frac{n_i}{n_t}\right)^2 \sin^2\theta_i - 1}$$

$$= \frac{632\text{nm}}{2\pi} \Big/ \sqrt{(1.5)^2 \sin^2\theta_i - 1}$$

At $\theta_i = 60°$ $\quad 1/K = 121\text{nm}$

$\theta_i = 75°$ $\quad 1/K = 96\text{nm}$

$\theta_i = 45°$ $\quad 1/K = 284\text{nm}$