HW 9 due today

HW 10 due Friday

Exam Monday

The Fresnel equations relate electric fields across an interface. For the field components:

$$E_{or}^{\perp} = r^{\perp} E_{oi}^{\perp} \qquad E_{or}^{\parallel} = r^{\parallel} E_{oi}^{\parallel}$$

$$E_{ot}^{\perp} = t^{\perp} E_{oi}^{\perp} \qquad E_{ot}^{\parallel} = t^{\parallel} E_{oi}^{\parallel}$$

where

$$r^{\perp} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} \qquad r^{\parallel} = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

$$t^{\perp} = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t} \qquad t^{\parallel} = \frac{2 n_i \cos\theta_i}{n_t \cos\theta_i + n_i \cos\theta_t}$$

We aim to convert these into statements about a more readily observable quantity, the irradiance of intensity of the beam. Recall that

irradiance = power transmitted by the beam per unit area.
$= \langle S \rangle$

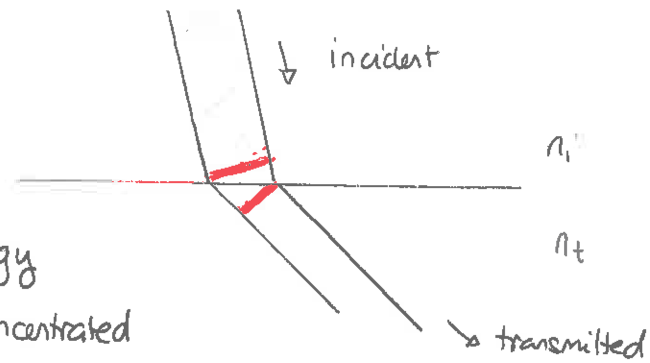
For harmonic plane waves in a medium with index of refraction n and described via $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ we showed that

$$\boxed{I = \frac{1}{2} \epsilon_0 c \, n \, \vec{E}_0 \cdot \vec{E}_0}$$

Reflectivity + Transmissivity

Consider light passing from a slower into a faster medium ($n_i > n_t$). We saw that $t^{\perp} > 1$ and also $t^{\parallel} > 1$ whenever $\theta_i < \theta_c$. How can the electric field be "amplified"?

If we consider beams of non-zero width then we see that the width of the transmitted beam is narrower that the incident beam. Thus the energy transported by the transmitted beam is concentrated into a smaller cross-sectional area. The intensity must therefore be larger. Denoting the intensity of the incident beam by I_i and the transmitted by I_t we can form:



$$\frac{I_t}{I_i} = \frac{\frac{1}{2}\epsilon_0 c\, n_t\, E_{0t}^2}{\frac{1}{2}\epsilon_0 c\, n_i\, E_{0i}^2} \Rightarrow \frac{I_t}{I_i} = \frac{n_t}{n_i}\left(\frac{E_{0t}}{E_{0i}}\right)^2$$

$$>1 \qquad <1 \qquad \Rightarrow \left(E_{0t}/E_{0i}\right)^2 > 1$$

It follows that $|E_{0t}| > |E_{0i}|$.

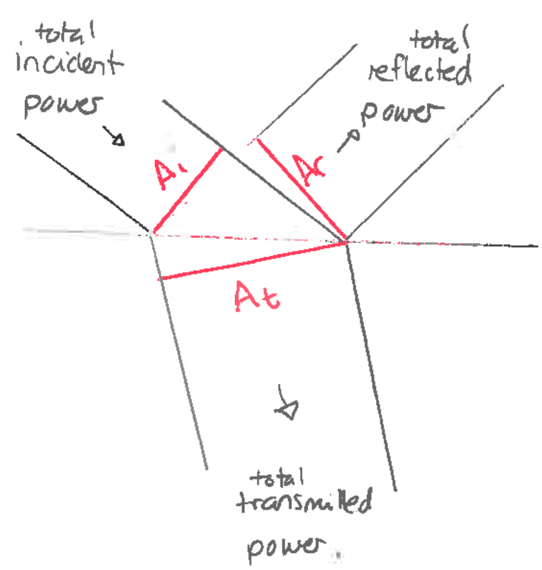
Cross sectional area is clearly a factor here and to incorporate this, rather than compare ratios of intensities, we will compare ratios of powers.

We then define the reflectivity as:

$$\boxed{R = \frac{\text{reflected power}}{\text{incident power}}}$$

and the transmittivity as:

$$\boxed{T = \frac{\text{transmitted power}}{\text{incident power.}}}$$



Using the illustrated cross-sectional areas

reflected power $= A_r I_r$

transmitted power $= A_t I_t$

incident power $= A_i I_i$

Thus

$$R = \frac{I_r}{I_i}\frac{A_r}{A_i}$$

$$T = \frac{I_t}{I_i}\frac{A_t}{A_i}$$

<u>Exercise:</u> a) Determine expressions for R, T for perpendicular polarization in terms of $n_i, n_t, \theta_i, \theta_t$.

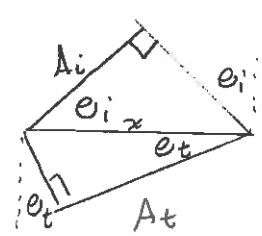
b) Show that $R+T=1$.

<u>Answer:</u> a) For reflection, symmetry gives $A_r = A_i$

$$\Rightarrow \quad R = \frac{I_r}{I_i} = \frac{\frac{1}{2}c\epsilon_0 n_i E_{0r}^2}{\frac{1}{2}c\epsilon_0 n_i E_{0i}^2} = r^2$$

$$\Rightarrow \quad R = \left(\frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_i\cos\theta_t}\right)^2$$

For transmission,



$$\frac{A_i}{x} = \cos\theta_i$$

$$\frac{A_t}{x} = \cos\theta_t$$

$$\Rightarrow \quad \frac{A_t}{A_i} = \frac{\cos\theta_t}{\cos\theta_i}$$

Thus

$$T = \frac{\cancel{\tfrac{1}{2} c \epsilon_0} n_t E_{ot}^2}{\cancel{\tfrac{1}{2} c \epsilon_0} n_i E_{oi}^2} \frac{\cos\theta_t}{\cos\theta_i}$$

$$= \frac{n_t}{n_i} \frac{\cos\theta_t}{\cos\theta_i} \left(\frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t} \right)^2$$

$$= 4 n_i n_t \frac{\cos\theta_t \cos\theta_i}{(n_i \cos\theta_i + n_t \cos\theta_t)^2}$$

b) $$R+T = \left[(n_i \cos\theta_i - n_t \cos\theta_t)^2 + 4 n_i n_t \cos\theta_t \cos\theta_i \right] \Big/ \left[n_i \cos\theta_i + n_t \cos\theta_t \right]^2$$

$$= \left[n_i^2 \cos^2\theta_i - 2 n_i n_t \cos\theta_i \cos\theta_t + n_t^2 \cos^2\theta_t + 4 \ldots \right] \Big/ \left[\ldots \right]^2$$

$$= (n_i \cos\theta_i + n_t \cos\theta_t)^2 \Big/ \left[\ldots \right]^2 = 1 \qquad \blacksquare$$

Since both $R, T \geq 0$ and $R+T=1$ we now have $0 < R, T \leq 1$. These are for perpendicular polarization. Thus we denote them with "$\perp$" giving

$$\boxed{R^\perp = \left[\frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} \right]^2 \qquad T^\perp = \frac{4 n_i n_t \cos\theta_t \cos\theta_i}{\left[n_i \cos\theta_i + n_t \cos\theta_t \right]}}$$

Similarly with parallel polarization, we get:

$$\boxed{R^{\parallel} = \left[\frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_t\cos\theta_i + n_i\cos\theta_t}\right]^2 \qquad T^{\parallel} = \frac{4n_in_t\cos\theta_t\cos\theta_i}{[n_t\cos\theta_i + n_i\cos\theta_t]^2}}$$

and we can show:

$$\boxed{R^{\parallel} + T^{\parallel} = 1 \quad \text{and} \quad R^{\perp} + T^{\perp} = 1}$$

So for these are valid only when $0 \leq \theta_i \leq \theta_c$.

<u>Incident in faster medium $n_i < n_t$</u>

In this case the equations work for all $0 \leq \theta_i \leq \pi/2$. We were able to show that $r^{\perp}$ is negative and decreases as θ_i increases. Now $R^{\perp}$ is positive and this will increase as θ_i increases. We can chart it exactly using:

$$r^{\perp} = \frac{\sqrt{1-\sin^2\theta_i} - \sqrt{\left(\frac{n_t}{n_i}\right)^2 - \sin^2\theta_i}}{\sqrt{1-\sin^2\theta_i} + \sqrt{\left(\frac{n_t}{n_i}\right)^2 - \sin^2\theta_i}}$$

$$\Rightarrow r^{\perp 2} = \left[\frac{\sqrt{\left(\frac{n_t}{n_i}\right)^2 - \sin^2\theta_i} - \sqrt{1-\sin^2\theta_i}}{\sqrt{\left(\frac{n_t}{n_i}\right)^2 - \sin^2\theta_i} + \sqrt{1-\sin^2\theta_i}}\right]^2$$

$$\Rightarrow R^{\perp} = \left[\qquad \text{this} \qquad \right]^2$$

and $T = 1 - R$ allows us to complete the plots.

Exercise: For $n_i < n_t$ and perpendicular polarization, determine the extreme values (over θ_i) of R and T.

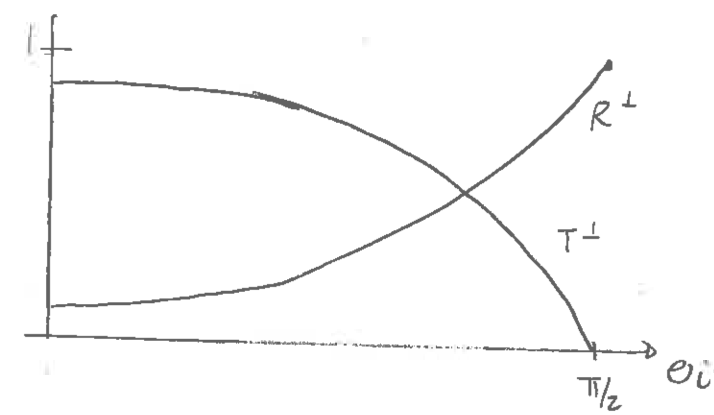
Answer: Smallest value occurs when $\theta_i = 0$

$$\Rightarrow R^{\perp} = \left[\frac{\frac{n_t}{n_i} - 1}{\frac{n_t}{n_i} + 1}\right]^2 = \frac{(n_t - n_i)^2}{(n_t + n_i)^2} \qquad \Rightarrow T = 1 - R = \frac{4 n_i n_t}{(n_t + n_i)^2}$$

Largest value occurs when $\theta_i = 90°$

$$R^{\perp} = 1 \qquad \Rightarrow \qquad T^{\perp} = 0$$

Plotting gives



Note that as $\frac{n_t}{n_i} \gg 1$ $\quad R \to 1$ for all angles and $T \to 0$ for all angles.

So in this extreme all power is reflected. Then as $n_t/n_i \to 1$ $R \to 0$ and $T \to 1$ for all angles except $\theta_i = 90°$.

Consider now, parallel polarization. Then:

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t}$$

but $\sin\theta_t = \frac{n_i}{n_t}\sin\theta_i \Rightarrow \cos\theta_t = \sqrt{1-\left(\frac{n_i}{n_t}\right)^2 \sin^2\theta_i}$

$$= \frac{n_i}{n_t}\sqrt{\left(\frac{n_t}{n_i}\right)^2 - \sin^2\theta_i}$$

Thus $$R^{\parallel} = \left[\frac{\frac{n_t}{n_i}\sqrt{1-\sin^2\theta_i} - \frac{n_i}{n_t}\sqrt{\left(\frac{n_t}{n_i}\right)^2-\sin^2\theta_i}}{\frac{n_t}{n_i}\sqrt{\quad} + \cdots}\right]^2$$

$$R^{\parallel} = \left[\frac{\left(\frac{n_t}{n_i}\right)^2\sqrt{1-\sin^2\theta_i} - \sqrt{\left(\frac{n_t}{n_i}\right)^2-\sin^2\theta_i}}{\left(\frac{n_t}{n_i}\right)^2\sqrt{1-\sin^2\theta_i} + \sqrt{\left(\frac{n_t}{n_i}\right)^2-\sin^2\theta_i}}\right]^2$$

and $T^{\parallel} = 1 - R^{\parallel}$

<u>Exercise</u> Determine when $R^{\parallel} = 0$

<u>Answer:</u> $$\left(\frac{n_t}{n_i}\right)^2\sqrt{1-\sin^2\theta_i} = \sqrt{\left(\frac{n_t}{n_i}\right)^2-\sin^2\theta_i}$$

$$\Rightarrow \left(\frac{n_t}{n_i}\right)^2\cos\theta_i = \sqrt{\left(\frac{n_t}{n_i}\right)^2-\sin^2\theta_i}$$

$$\Rightarrow \left(\frac{n_t}{n_i}\right)^4\cos^2\theta_i = \left(\frac{n_t}{n_i}\right)^2 - \sin^2\theta_i$$

$$\Rightarrow \left(\frac{n_t}{n_i}\right)^4(1-\sin^2\theta_i) = \left(\frac{n_t}{n_i}\right)^2 - \sin^2\theta_i$$

$$\Rightarrow \left(\frac{n_t}{n_i}\right)^2\left[\left(\frac{n_t}{n_i}\right)^2-1\right] = \sin^2\theta_i\left[\left(\frac{n_t}{n_i}\right)^4-1\right] \rightarrow \left[\left(\frac{n_t}{n_i}\right)^2-1\right]\left[\left(\frac{n_t}{n_i}\right)^2+1\right]$$

$$\Rightarrow \left(\frac{n_t}{n_i}\right)^2 = \sin^2\theta_i\left[\left(\frac{n_t}{n_i}\right)^2+1\right]$$

$$\Rightarrow \left(\frac{n_t}{n_i}\right)^2\left[1-\sin^2\theta_i\right] = \sin^2\theta_i$$

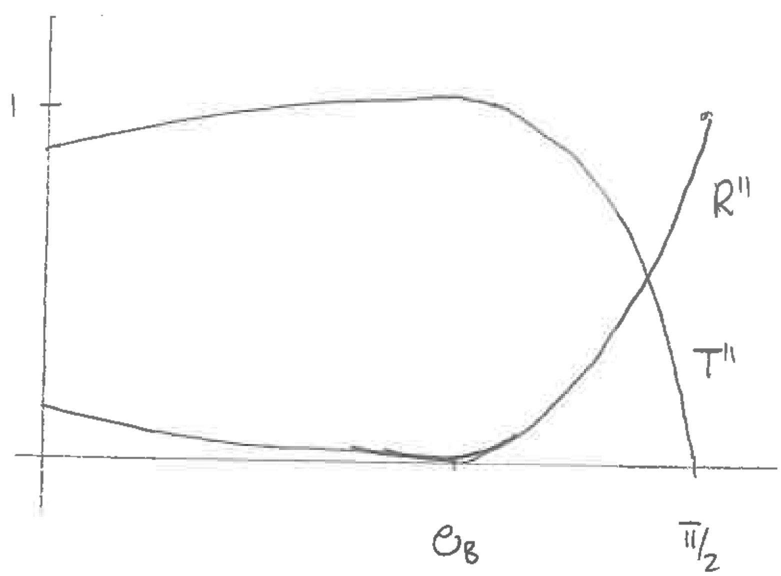
$$\Rightarrow \left(\frac{n_t}{n_i}\right)^2 \cos^2\theta_i = \sin^2\theta_i$$

$$\Rightarrow \frac{\sin\theta_i}{\cos\theta_i} = \frac{n_t}{n_i} \quad \Rightarrow \quad \tan\theta_i = \frac{n_t}{n_i}$$

This is the condition for Brewster's angle:

$$\theta_B = \tan^{-1}\left(\frac{n_t}{n_i}\right)$$

We can show that R'' decreases with θ_i from $\theta_i=0$ to $\theta_i=\theta_B$ and it increases from θ_B to $\pi/2$. Plotting gives



Exercise: Suppose $n_t \gg n_i$. Then

a) find the approximate value of Brewster's angle

b) approximate $R^{\parallel}$ and $T^{\parallel}$

Answer: a) $\theta_B = \tan^{-1}\left(\frac{n_t}{n_i}\right) \rightarrow \tan^{-1}(\infty) = \pi/2$

b)
$$R^{\parallel} \approx \left[\frac{\left(\frac{n_t}{n_i}\right)^2\sqrt{1-\sin^2\theta_i} - \frac{n_t}{n_i}}{\left(\frac{n_t}{n_i}\right)^2\sqrt{1-\sin^2\theta_i} + \frac{n_t}{n_i}}\right]^2$$

$$= \left[\frac{\frac{n_t}{n_i}\sqrt{1-\sin^2\theta_i} - 1}{\frac{n_t}{n_i}\sqrt{1+\sin^2\theta_i} + 1}\right]^2$$

$$= \left[\frac{\sqrt{1-\sin^2\theta_i} - \frac{n_i}{n_t}}{\sqrt{1-\sin^2\theta_i} + \frac{n_i}{n_t}}\right]^2$$

As $\theta_i \rightarrow 90^\circ$ this approaches 1

As $\theta_i \rightarrow 0^\circ$ this approaches $\left[\frac{1-n_i/n_t}{1+n_i/n_t}\right]^2 \approx 1$