Phys 473 2015

Lecture 11

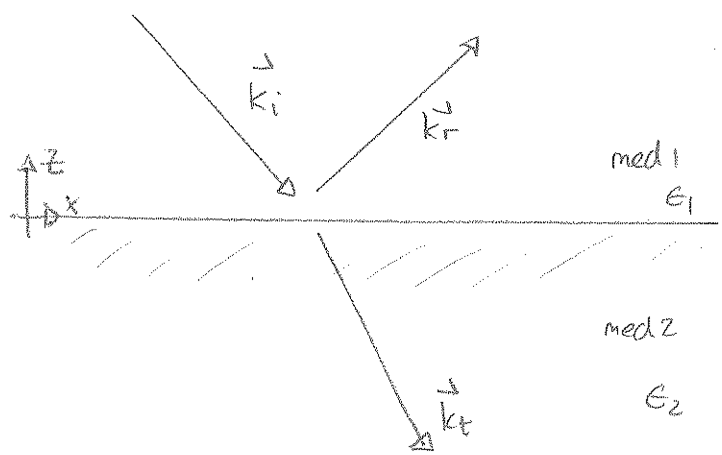
<u>Weds</u> HW 7 due today

<u>Thurs</u> Seminar 12:30 WS 117

<u>Fri:</u> HW 8 due.

## Electromagnetic Waves at an Interface

We consider the scattering of a plane harmonic wave on a plane surface. This produces a reflected wave ["r"] and a transmitted wave ["t"]. We found that




1) the angular frequencies of all three waves are identical.

2) the wave vectors for the three waves all lie in the same plane - called the <u>plane of incidence</u>

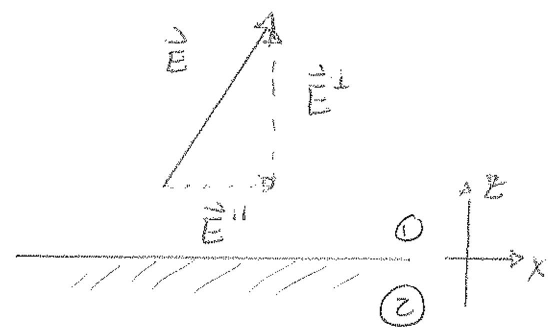
The plane of incidence is perpendicular to the surface that constitutes the interface. In the illustration the plane of incidence is the same as the plane of the page. Then we describe the waves via:

incident: $\vec{E}_i = \vec{E}_{oi}\, e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$

reflected: $\vec{E}_r = \vec{E}_{or}\, e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$

transmitted: $\vec{E}_t = \vec{E}_{ot}\, e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$

When applying the boundary conditions that result from Maxwell's equations, it is useful to decompose and electric field vector into vectors parallel and perpendicular to the surface. Using the illustrated co-ordinates we have




$$\vec{E} = \vec{E}^{\parallel} + \vec{E}^{\perp}$$

$\vec{E}^{\perp}$ → component in z direction

$\vec{E}^{\parallel}$ → components in x, y directions

With this system of co-ordinates the boundary conditions result in:

$$E_{oix} + E_{orx} = E_{otx}$$

$$E_{oiy} + E_{ory} = E_{oty}$$

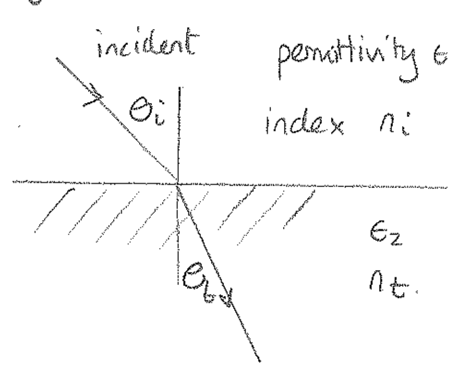
$$E_{oiz} + E_{orz} = \frac{\epsilon_2}{\epsilon_1} E_{otz}$$

$$E_{oiy} - E_{ory} = \frac{n_t}{n_i}\frac{\cos\theta_t}{\cos\theta_i} E_{oty}$$

$$\sin\theta_i\left(E_{oiz} + E_{orz}\right) + \cos\theta_i\left(E_{orx} - E_{oix}\right) = \frac{n_t}{n_i}\left(\sin\theta_i E_{otz} + \cos\theta_t E_{otx}\right)$$

The symbols are illustrated in the accompanying diagram

This system of equations allows us to treat the following separately




i) All y-components i.e. components parallel to the boundary or perpendicular to plane of incidence

ii) All x-z components i.e. components parallel to plane of incidence.

Perpendicular components: perpendicular polarization.

Combining the equations for the $y$ components gives:

$$E_{ory} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} E_{oiy}$$

$$E_{oty} = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t} E_{oiy}$$

As these are the perpendicular components, we write them:

$$\boxed{\begin{aligned} E_{or}^{\perp} &= r^{\perp} E_{oi}^{\perp} \\ E_{ot}^{\perp} &= t^{\perp} E_{oi}^{\perp} \end{aligned}}$$

where the Fresnel amplitude ratios are:

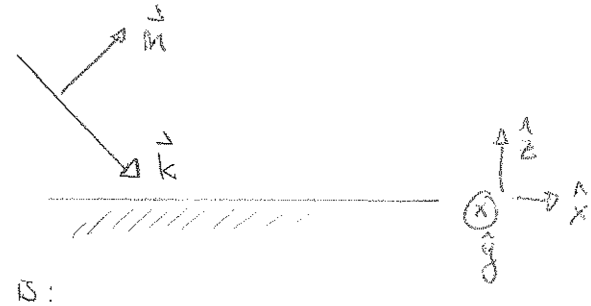
$$\boxed{r^{\perp} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}}$$

and

$$\boxed{t^{\perp} = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}}$$

## Parallel components

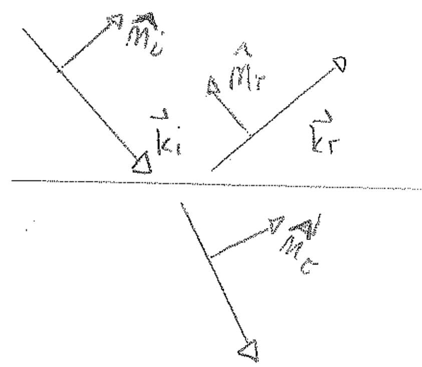
Consider the equations for $x, z$ components. There are three equations relating six variables and we could try to solve these using linear algebra. The mathematics can be vastly simplified by reorganizing the vector components. We know that any electric field is perpendicular to the $\vec{k}$ vector. If it also lies in the plane of incidence then it only has a non-zero component along the vector $\hat{m}$. Thus the parallel polarization field is:




$$\vec{E} = E_m \hat{m}$$

and we can always extract $E_m$ by

$$E_m = \hat{m} \cdot \vec{E}$$




We can do this construction using a different $\hat{m}$ vector for each wave. These are always constructed via $\hat{m} = \hat{k} \times \hat{y}$

$\hat{k}$ ↳ unit vector along $\vec{k}$

So for the incident wave $\vec{E}_i = \vec{E}_{0i}\, e^{i(\ldots)}$ results in:

$$E_{0im} = \vec{E}_{0i} \cdot \hat{m}_i$$

and similarly for the other waves. Then.

$$\vec{E}_i = E_{0im}\, e^{i(\ldots)}\, \hat{m}_i$$

and all we need to describe the fields are the scalar $E_{0im}, E_{0rm}, E_{0tm}$.

We label each as $E_{0i}{}^{\parallel}, E_{0r}{}^{\parallel}$

We shall show that these satisfy:

$$\boxed{\begin{aligned} E_{oi}'' + E_{or}'' &= \frac{n_t}{n_i} E_{ot}'' \\ E_{oi}'' - E_{or}'' &= \frac{\cos\theta_t}{\cos\theta_i} E_{ot}'' \end{aligned}}$$

These provide a complete set of relationships.

<u>Proof:</u> By construction $\hat{m} = \hat{k} \times \hat{y}$ and $E_m = \vec{E} \cdot (\hat{k} \times \hat{y})$

$$= -\hat{y} \cdot (\hat{k} \times \vec{E})$$

But $\vec{k} \times \vec{E} = \omega \vec{B} \Rightarrow \hat{k} \times \vec{E} = \frac{\omega}{k} \vec{B} = V \vec{B}$

$$\Rightarrow \hat{k} \times \vec{E} = c/n \, \vec{B}.$$

$$\Rightarrow E_m = -c/n \, B_y \Rightarrow -\frac{n}{c} E_m = B_y$$

Continuity of magnetic fields gives:

$$B_{oiy} + B_{ory} = B_{oty} \Rightarrow \frac{n_i}{c} E_{oim} + \frac{n_i}{c} E_{orm} = \frac{n_t}{c} E_{otm}$$

$$\Rightarrow E_{oim} + E_{orm} = \frac{n_t}{n_i} E_{otm}$$

and this gives the first equation

Separately at the boundary $E_{oix} + E_{oix} = E_{otx}$. Then

$$E_x = \vec{E} \cdot \hat{x} = E_m \, \hat{m} \cdot \hat{x} = E_m (\hat{k} \times \hat{y}) \cdot \hat{x} = E_m \, \hat{x} \cdot (\hat{k} \times \hat{y}) = E_m \hat{k} (\hat{y} \times \hat{x})$$

$$= E_m \, \hat{k} \cdot (-\hat{z})$$


$$\text{Now} \quad \left.\begin{aligned} \hat{k}_i \cdot (-\hat{z}) &= \cos\theta_i \\ k_t \cdot (-\hat{z}) &= \cos\theta_t \\ \hat{k}_r \cdot (-\hat{z}) &= -\cos\theta_r \end{aligned}\right\} \Rightarrow \begin{aligned} &E_{oim} \cos\theta_i - E_{orm} \cos\theta_r \\ &= E_{otm} \cos\theta_t \end{aligned}$$

and $\theta_i = \theta_r$ gives the result ∎

Exercise: 1: Rewrite the equations for parallel components to give:

$$E_{or}^{\parallel} = \ldots$$

$$E_{ot}^{\parallel} = \ldots$$

Answer: Adding gives:
$$2E_{oi}^{\parallel} = \left[\frac{n_t}{n_i} + \frac{\cos\theta_t}{\cos\theta_i}\right] E_{ot}^{\parallel}$$

$$= \frac{n_t\cos\theta_i + n_i\cos\theta_t}{n_i\cos\theta_i} E_{ot}^{\parallel}$$

$$\Rightarrow \quad E_{ot}^{\parallel} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i} E_{oi}^{\parallel}$$

Subtracting gives:
$$2E_{or}^{\parallel} = \left[\frac{n_t}{n_i} - \frac{\cos\theta_t}{\cos\theta_i}\right] E_{ot}^{\parallel}$$

$$= \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_i} E_{ot}^{\parallel}$$

$$= \frac{2\left[n_t\cos\theta_i - n_i\cos\theta_t\right]}{n_i\cos\theta_t + n_t\cos\theta_i} E_{oi}^{\parallel}.$$

$$\Rightarrow \quad E_{or}^{\parallel} = \frac{[\ldots]}{[\ldots]} E_{oi}^{\parallel}$$

Thus we find the Fresnel equations for parallel polarization:

$$\boxed{\begin{array}{ll} E_{or}^{\parallel} = r^{\parallel} E_{oi}^{\parallel} & r^{\parallel} = \dfrac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i} \\ E_{ot}^{\parallel} = t^{\parallel} E_{oi}^{\parallel} & t^{\parallel} = \dfrac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i} \end{array}}$$

# Applying the Fresnel equations

The Fresnel equations show that reflection + transmission processes strongly depend on the polarization.

Exercise: For perpendicular polarization, determine:

a) when there is no transmission

b) when " " " reflection.

c) the value of the other coefficient in each case:

Answer: a) $t^{\perp} = 0 \Rightarrow \cos\theta_i = 0 \Rightarrow \theta_i = 90°$. Then

$r^{\perp} = -1$ (all "reflected")

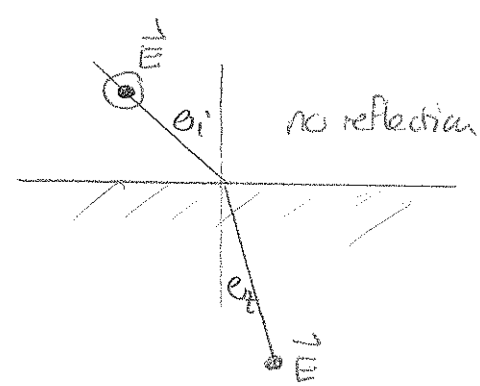
b) $n_i \cos\theta_i = n_t \cos\theta_t$.

$t^{\perp} = 1$ by calculation (all transmitted).

In the last case we get no reflection. Snell's law can give a rule for this

Exercise: Use Snell's law to show that this only occurs when

$n_i = n_t$



Answer:  $n_i \cos\theta_i = n_t \cos\theta_t$

But  $n_i \sin\theta_i = n_t \sin\theta_t$.

$\Rightarrow \quad \tan\theta_i = \tan\theta_t \Rightarrow \quad \theta_i = \theta_t$

This is only possible when  $n_i = n_t$

Thus:

> For perpendicular polarization, the only situation in which perfect transmission occurs is when $n_i = n_t$.