Phys 473 2015

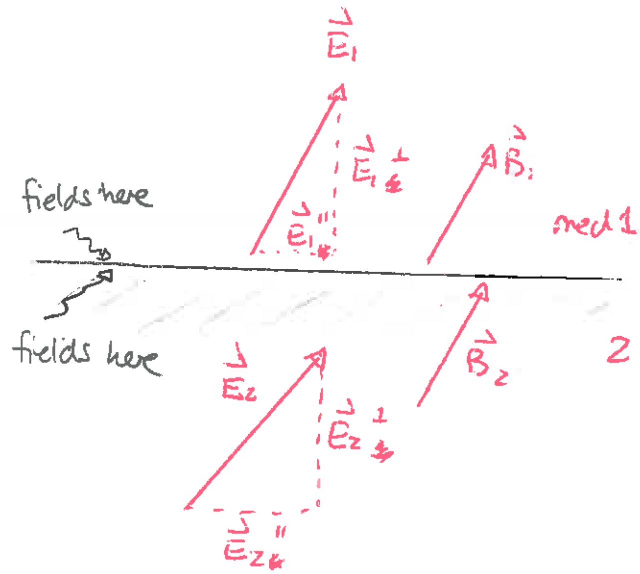
Lecture 10

HW 7 due Weds

SPS Meeting

Boundary conditions for electromagnetic fields

Maxwell's equations provide general matching conditions for electric and magnetic fields on either side of a boundary. For dielectric optical materials which are not magnetically active these reduce to:



$$\boxed{\begin{aligned} \vec{B}_1 &= \vec{B}_2 \\ \vec{E}_1^{\parallel} &= \vec{E}_2^{\parallel} \\ \epsilon_1 \vec{E}_1^{\perp} &= \epsilon_2 \vec{E}_2^{\perp} \end{aligned}}$$

where the electric field is decomposed into a two components:

$$\vec{E}_1 = \vec{E}_1^{\parallel} + \vec{E}_1^{\perp}$$

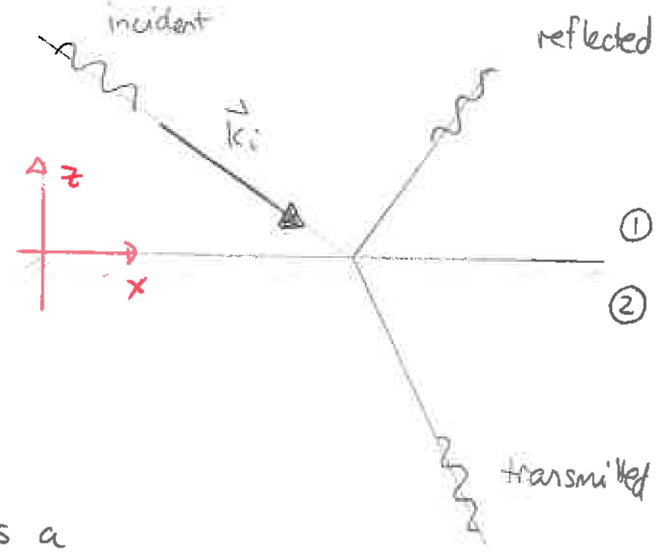
parallel to surface — perpendicular to surface.

Here ϵ_1 = permittivity of medium 1

ϵ_2 = " " " 2

These will govern the reflection and transmission of waves at the interface.

We will consider a plane harmonic incident wave and will show that it is possible to satisfy the bondary conditions by assuming that this wave generates a reflected and also a transmitted wave. Each wave can be described via a complex representation of the form



$$\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

where $\vec{E}_0$ might be complex and $\vec{k}$ is a wave vector and ω an angular frequency

Description of all waves

The three waves are described by

Incident wave: $\vec{E}_i = \vec{E}_{oi} \, e^{i(\vec{k}_i\cdot\vec{r} - \omega_i t)}$ — wave vector $\vec{k}_i$, ang. freq ω_i

Reflected wave: $\vec{E}_r = \vec{E}_{or} \, e^{i(\vec{k}_r\cdot\vec{r} - \omega_r t)}$ — wavevector $\vec{k}_r$, freq ω_r

Transmitted wave $\vec{E}_t = \vec{E}_{ot} \, e^{i(\vec{k}_t\cdot\vec{r} - \omega_t t)}$ — " $\vec{k}_t$ " ω_t

For each we have: $\omega = kv = kc/n$ and thus

$$\omega_i = k_i c/n_i$$

$$\omega_r = k_r c/n_r$$

$$\omega_t = k_t c/n_t$$

where n is the relevant index of refraction. The associated magnetic fields are generated via:

$$\vec{B}_i = \frac{1}{\omega_i}(\vec{k}_i \times \vec{E}_i) \quad \text{etc.,..}$$

Now wherever $\vec{r}$ refers to a location on the boundary the matching conditions between the field in medium ①,

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r = \vec{E}_{oi}\, e^{i(\vec{k}_i \cdot \vec{r} - \omega_i t)} + \vec{E}_{or}\, e^{i(\vec{k}_r \cdot \vec{r} - \omega_r t)}$$

and that in medium ②

$$\vec{E}_2 = \vec{E}_t = \vec{E}_{ot}\, e^{i(\vec{k}_t \cdot \vec{r} - \omega_t t)}$$

apply. We stress that the matching conditions do not necessarily apply beyond the boundary. We shall use these to:

1) relate frequencies
2) relate wave vectors
3) relate electric fields

There is one key mathematical rule:

> If $Ae^{\alpha x} + Be^{\beta x} = Ce^{\gamma x}$ holds for all x then
>
> i) $\alpha = \beta = \gamma$ and
>
> ii) $A + B = C$.

To prove this the first gives:

$$Ae^{(\alpha-\gamma)x} + Be^{(\beta-\gamma)x} = C$$

$$\Rightarrow \frac{d}{dx}(\ldots) = \frac{d}{dx}C = 0 \quad \Rightarrow \quad A(\alpha-\gamma)e^{(\alpha-\gamma)x} + B(\beta-\gamma)e^{(\beta-\gamma)x} = 0$$

$$\Rightarrow A(\alpha-\gamma)e^{\alpha x} + B(\beta-\gamma)e^{\beta x} = 0$$

Differentiate again

$$\Rightarrow A(\alpha-\gamma)e^{(\alpha-\beta)x} + B(\beta-\gamma) = 0$$

$$\Rightarrow A(\alpha-\gamma)(\alpha-\beta) = 0$$

So $\alpha = \gamma$ or $\alpha = \beta \Rightarrow$ one substitutes and reduces this to $Ae^{\alpha x} + Be^{\beta x} = 0$

The matching conditions at the boundary are:

$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \Rightarrow \vec{E}_{oi}^{\parallel} e^{i(\vec{k}_i\cdot\vec{r} - \omega_i t)} + \vec{E}_{or}^{\parallel} e^{i(\vec{k}_r\cdot\vec{r} - \omega_r t)} = \vec{E}_{ot}^{\parallel} e^{i(\vec{k}_T\cdot\vec{r} - \omega_T t)}$$

$$\epsilon_1 \vec{E}_1^{\perp} = \epsilon_2 \vec{E}_2^{\perp} \Rightarrow \epsilon_1 \vec{E}_{oi}^{\perp} e^{i(\vec{k}_i\cdot\vec{r} - \omega_i t)} + \epsilon_1 \vec{E}_{or}^{\perp} e^{i(\dots)} = \epsilon_2 \vec{E}_{ot}^{\perp} e^{i(\dots)}$$

Frequencies.

At least one of these must contain a non-zero r.h.s. Suppose it is the first. Then fix $\vec{r}$ and this takes the form:

$$\vec{A} e^{-i\omega_i t} + \vec{B} e^{-i\omega_r t} = \vec{C} e^{-i\omega_t t}$$

By the previous theorem this is only possible if

$$\boxed{\begin{array}{l}\text{The angular frequencies are all equal} \\ \Rightarrow \omega_i = \omega_r = \omega_t \equiv \omega\end{array}}$$

Wave vectors.

We shall use a co-ordinate system which is such that

i) the surface is in the $z=0$ plane (or xy plane)

ii) the incident wave vector is in the (xz plane)

Then when $z=0$ the matching conditions above must hold. Thus we get an equation of the form

$$\vec{A} e^{i(k_{ix}x + k_{iy}y - \omega t)} + \vec{B} e^{i(k_{rx}x + k_{ry}y - \omega t)} = \vec{C} e^{i(k_{tx}x + k_{ty}y - \omega t)}$$

This can only be true if it is true for all x and this gives:

$$k_{ix} = k_{rx} = k_{tx}$$

$$k_{iy} = k_{ry} = k_{ty}$$

But $k_{iy} = 0 \Rightarrow$

$$\vec{k}_i = k_{ix}\hat{\imath} + k_{iz}\hat{k}$$

$$\vec{k}_r = k_{rx}\hat{\imath} + k_{rz}\hat{k}$$

$$\vec{k}_t = k_{tx}\hat{\imath} + k_{tz}\hat{k}$$

These all lie in the x-z plane. So we have:

> The directions of propagation of the three waves are all in the same plane. This plane is perpendicular to the surface and is called the <u>plane of incidence</u>.

One can find the plane of incidence by:

i) find all planes perpendicular to the surface

ii) " " " in which $\vec{k}_i$ lies.

iii) only one pair coincides and this is the plane of incidence.

We then see:

Exercise: Use the relation $\omega = kv$ for each of these to:

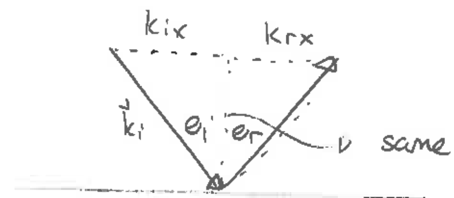
a) relate $\vec{k}_r$ to $\vec{k}_i$ and obtain the law of reflection

b) relate $\vec{k}_t$ to $\vec{k}_i$ and obtain Snell's Law.

Answer: a) $\omega_i = k_i v_i$ and $\omega_r = k_r v_r$. But $v_i = v_r$ and $\omega_i = \omega_r \Rightarrow k_i = k_r$.

Thus $k_{rz} = -k_{iz}$.

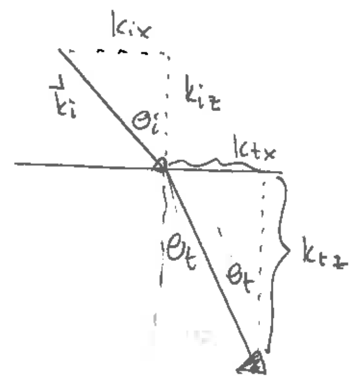
and the diagram clearly gives $\theta_i = \theta_r$



b) $\omega_t = k_t v_t$ and $\omega_i = k_i v_i$ with $\omega_t = \omega_i$ gives

$$k_t v_t = k_i v_i \Rightarrow k_t \frac{c}{n_t} = k_i \frac{c}{n_i}$$

$$\Rightarrow \frac{k_t}{n_t} = \frac{k_i}{n_i}$$



But $k_{tx} = k_{ix}$

$$\Rightarrow k_t \sin\theta_t = k_i \sin\theta_i$$

$$\Rightarrow k_i \frac{n_t}{n_i} \sin\theta_t = k_i \sin\theta_i$$

$$\Rightarrow n_t \sin\theta_t = n_i \sin\theta_i$$

Relating electric fields.

We now aim to relate electric fields at the boundary. These will eventually provide rules relating the intensities of the three waves. Recall that the propagation directions are related via:

$$\vec{k}_i = k_{ix}\,\hat{\imath} + k_{iz}\,\hat{k}$$

$$\vec{k}_r = k_{ix}\,\hat{\imath} - k_{iz}\,\hat{k}$$

$$\vec{k}_t = k_{ix}\,\hat{\imath} + k_{tz}\,\hat{k}$$

where

$$k_{ix} = k_i \sin\theta_i \qquad k_{tx} = k_t \sin\theta_t$$

$$k_{iz} = -k_i \cos\theta_i \qquad k_{tz} = k_t \cos\theta_t \qquad k_t = k_i \frac{n_t}{n_i}$$

Then the matching conditions yield:

$$\boxed{\begin{aligned}
&E_{oix} + E_{orx} = E_{otx}\\
&E_{oiy} + E_{ory} = E_{oty}\\
&E_{oiz} + E_{orz} = \frac{\epsilon_2}{\epsilon_1} E_{otz}\\
&E_{oiy} - E_{ory} = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} E_{oty}\\
&\sin\theta_i\left(E_{oiz}+E_{orz}\right) - \cos\theta_i\left(E_{orx}-E_{oix}\right) = \frac{n_t}{n_i}\left(-\sin\theta_i E_{otz} + \cos\theta_t E_{otx}\right)
\end{aligned}}$$

<u>Proof</u>. i) $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \Rightarrow E_{1x} = E_{2x}$ AND $E_{1y} = E_{2y}$

This yield the first two equations.

ii) $\vec{E}_1^{\perp} = \frac{\epsilon_2}{\epsilon_1} \vec{E}_2^{\perp} \Rightarrow E_{1z} = \frac{\epsilon_2}{\epsilon_1} E_{2z}$ yields the third equation.

iii) $\vec{B}_1 = \vec{B}_2$

$$\Rightarrow \vec{B}_i + \vec{B}_r = \vec{B}_t$$

$$\Rightarrow \cancel{\frac{1}{\omega}}(\vec{k}_i \times \vec{E}_i) + \cancel{\frac{1}{\omega}}(\vec{k}_r \times \vec{E}_r) = \cancel{\frac{1}{\omega}}(\vec{k}_t \times \vec{E}_t)$$

$$\Rightarrow \left[k_{ix}\hat{i} + k_{iz}\hat{k}\right] \times \left[E_{oix}\hat{i} + E_{oiy}\hat{j} + E_{oiz}\hat{k}\right]$$

$$+ \left[k_{ix}\hat{i} - k_{iz}\hat{k}\right] \times \left[E_{orx}\hat{i} + E_{ory}\hat{j} + E_{orz}\hat{k}\right]$$

$$= \left[k_{ix}\hat{i} + k_{tz}\hat{k}\right] \times \left[E_{otx}\hat{i} + E_{oty}\hat{j} + E_{otz}\hat{k}\right]$$

Comparing $\hat{i}$ components gives:

$$k_{iz}E_{oiy} - k_{iz}E_{ory} = k_{tz}E_{oty} \Rightarrow E_{oiy} - E_{ory} = \frac{k_{tz}}{k_{iz}} E_{oty}$$

$$= \frac{k_t}{k_i}\frac{\cos\theta_t}{\cos\theta_i} E_{oty}$$

which gives the fourth equation.

Comparing $\hat{k}$ components gives:

$$k_{ix}E_{oiy} + k_{ix}E_{ory} = k_{ix}E_{oty}$$ redundant (second)

Comparing $\hat{j}$ components $\Rightarrow$

$$k_{ix}E_{oiz} - k_{iz}E_{oix} + k_{ix}E_{orz} + k_{iz}E_{orx} = +k_{tz}E_{otx} - k_{ix}E_{otz}$$

$$\Rightarrow k_{ix}(E_{oiz} + E_{orz}) + k_{iz}(E_{orx} - E_{oix}) = k_i\frac{n_t}{n_i}\cos\theta_t E_{otx} - k_i\sin\theta_i E_{otz}$$

$$\Rightarrow k_i\sin\theta_i(\ldots) - k_i\cos\theta_i(\ldots) = k_i\frac{n_t}{n_i}\cos\theta_i E_{otx} - k_i\sin\theta_i E_{otz}$$

These state that we can treat the y component (perpendicular to plane of incidence) separately from the xz component (parallel to plane of incidence).

Perpendicular component

Suppose that the parallel components are zero. Then the only components of the $\vec{E}$-field are along y and the waves are all linearly polarized parallel to the surface. Then:

$$E_{oiy} + E_{ory} = E_{oty}$$

$$E_{oiy} - E_{ory} = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} E_{oty}$$

Exercise: Solve these for E_{ory} and E_{oty} in terms of E_{oi}

Answer: Adding gives:

$$2E_{oiy} = \left[\frac{n_t \cos\theta_t}{n_i \cos\theta_i} + 1\right] E_{oty}$$

$$\Rightarrow \quad E_{oty} = \left[\frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}\right] E_{oiy}$$

Subtracting gives:

$$2E_{ory} = E_{oty}\left[1 - \frac{n_t \cos\theta_t}{n_i \cos\theta_i}\right] = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}\left[1 - \dots\right] E_{oiy}$$

$$\Rightarrow \quad E_{ory} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} E_{oiy}$$

This gives, for the perpendicular polarization,

$$\boxed{\begin{aligned} E_{ory} &= \left[\frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}\right] E_{oiy} \\ E_{oty} &= \left[\frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}\right] E_{oiy} \end{aligned}}$$