Phys 473 2015

Lecture 7

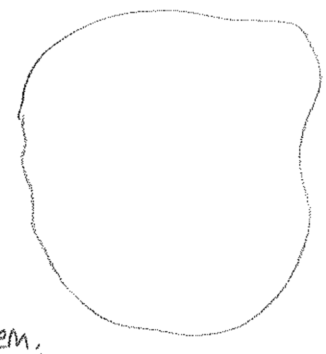
HW 5 on website

HW 6 by Friday

Energy flow in electromagnetism

The energy contained by electric field $\vec{E}$ and magnetic field $\vec{B}$ within any given region is:

$$U = \frac{1}{2}\int_{\text{region}} \left[\epsilon_0 \vec{E}\cdot\vec{E} + \frac{1}{\mu_0}\vec{B}\cdot\vec{B}\right] d\tau$$



We have seen examples in which the energy is clearly transported from one region to another. This can be quantified by the Poynting theorem.

Here consider a region in which there is no current. Then

The rate at which energy enters any closed region is:

$$\frac{dU}{dt} = -\oint_{\text{surface of region}} \vec{S}\cdot d\vec{a}$$

where

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

is the Poynting vector.

Proof:
$$\frac{\partial u}{\partial t} = \frac{1}{2}\int \Big[\epsilon_0 \frac{d\vec{E}}{dt}\cdot\vec{E} + \epsilon_0 \vec{E}\cdot\frac{d\vec{E}}{dt} + \frac{1}{\mu_0}\frac{d\vec{B}}{dt}\cdot\vec{B} + \frac{1}{\mu_0}\vec{B}\cdot\frac{d\vec{B}}{dt}\Big]\,d\tau$$

$$= \int \epsilon_0\left\{\frac{\partial\vec{E}}{\partial t}\cdot\vec{E} + \frac{1}{\mu_0}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}\right\}d\tau.$$

Now
$$\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$$
$$\frac{\partial\vec{E}}{\partial t} = \frac{1}{\mu_0\epsilon_0}\vec{\nabla}\times\vec{B}$$

gives:
$$\frac{\partial u}{\partial t} = \frac{1}{\mu_0}\int\Big[(\vec{\nabla}\times\vec{B})\cdot\vec{E} - \vec{B}\cdot(\vec{\nabla}\times\vec{E})\Big]\,d\tau.$$

Then $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$ can be applied to:
$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$

Thus
$$\frac{\partial u}{\partial t} = -\frac{1}{\mu_0}\int \vec{\nabla}\cdot(\vec{E}\times\vec{B})\,d\tau$$
$$= -\frac{1}{\mu_0}\oint \vec{E}\times\vec{B}\cdot d\vec{a}$$

By the divergence theorem.

Note that if $\vec{S}$ is pointing outward then $\vec{S}\cdot d\vec{a} > 0$ and this results in a negative $\frac{\partial u}{\partial t}$ meaning that energy leaves the region.

Thus we can interpret the <u>Poynting vector</u> as:

$\vec{S}$ describes the rate at which energy flows per unit area and per second. The direction of $\vec{S}$ describes the direction of energy flow.

To get the energy that passes across a surface in a given direction we do:

i) construct a vector in the direction of interest that is perpendicular to the surface

ii) Compute

$$\int_{\text{surface}} \vec{S}\cdot d\vec{a}$$

and this gives the energy that flows per second.

Exercise: Consider waves of the form:

$$\vec{E} = \vec{E}_0 f(u)$$

where $u = \vec{k}\cdot\vec{r} - \omega t + \varphi$ and f is any function. For this type of wave solution

$$\omega \vec{B} = \vec{k} \times \vec{E}$$

a) Determine a general expression for the Poynting vector for this type of wave, in terms of the electric field and ϵ_0

b) Consider a harmonic wave:

$$\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$

Determine the Poynting vector for this.

c) If $E_0 = 10\,N/C\ \hat{j}$ and the $\vec{k}$ vector is along $\hat{i}$ determine the energy that flows through a window of radius 5.0cm in the y-z plane per second

Answers: a)

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \frac{1}{\omega}(\vec{k}\times\vec{E})$$

$$= \frac{1}{\mu_0}\frac{1}{\omega}\vec{E}\times(\vec{k}\times\vec{E})$$

$$= \frac{1}{\mu_0}\frac{1}{\omega}\left\{\vec{k}(\vec{E}\cdot\vec{E}) - \vec{E}(\underbrace{\vec{k}\cdot\vec{E}}_{=0})\right\}$$

$$= \frac{\vec{k}}{\mu_0\omega}\vec{E}\cdot\vec{E}$$

Then $c = \omega/k \Rightarrow \omega = ck.$

$$\vec{S} = \frac{\vec{k}}{\mu_0 ck}\vec{E}\cdot\vec{E}$$

and $\mu_0\epsilon_0 = 1/c^2$ gives

$$\vec{S} = \epsilon_0 c\,\vec{E}\cdot\vec{E}\left(\frac{\vec{k}}{k}\right)$$

$\Rightarrow$ unit vector in direction of propagation

b) $$\vec{S} = \epsilon_0 c E_0^2\cos^2(\vec{k}\cdot\vec{r}-\omega t)\left(\frac{\vec{k}}{k}\right)$$

c) $\vec{k}/k = \hat{i} \Rightarrow$ $$\vec{S} = \epsilon_0 c E_0^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)\,\hat{i}$$

$$= 8.85\times10^{-12}\frac{C^2}{Nm^2} \times 3\times10^8 m/s \times (10 N/C)^2\cos^2(\ldots)\hat{i}$$

$$= 0.26\ \frac{N}{ms}\cos^2(\vec{k}\cdot\vec{r}-\omega t)\,\hat{i}$$

Now the rate at which energy flows is $\int \vec{S}\cdot d\vec{a}$

Here $\quad d\vec{a} = da\,\hat{\imath}$ and $\vec{S}\cdot d\vec{a} = 0.26\,\text{N}/\text{ms}\,\cos^2(kx-\omega t)$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = -\,0.26\,\frac{\text{N}}{\text{ms}}\cos^2(kx-\omega t)\times\text{area}$$

$$= 0.26\,\frac{\text{N}}{\text{ms}}\cos^2(kx-\omega t)\times\pi r^2$$

$$= 0.26\,\frac{\text{N}}{\text{ms}}\cos^2(kx-\omega t)\times\pi\,(0.05\,\text{m})^2$$

$$= -\,0.0021\,\text{W}\,\cos^2(kx-\omega t)$$

This is always positive but it does fluctuate with time.

Irradiance and Intensity

We have seen that the rate at which energy flows through any surface is given via the Poynting vector $\vec{S}$ and that for a harmonic electromagnetic wave of the form:

$$\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$

the Poynting vector is:

$$\vec{S} = \epsilon_0 c\, E_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t)\ \underbrace{\frac{\vec{k}}{k}}_{\text{unit vector in direction of flow.}}$$

The magnitude of the Poynting vector is:

$$S = \epsilon_0 c\, E_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t)$$

and this fluctuates with time and location. In optical situations the frequencies are large ($\sim 10^{14}$ Hz) and these oscillating energy fluctuations pass unnoticed. Rather, what we observe is an averaged rate of energy as time passes. The time average of any function of time is defined as:

$$\boxed{\langle f \rangle := \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt}$$

There are several ways to apply this to optical situations. Two possibilities are:

i) for a harmonic wave, or wave with one frequency, determine the average over one cycle.

ii) more generally determine the average over a time period that is large compared to typical wave oscillations but small enough to capture information about changes in wave amplitude.

Exercise 2: Consider the harmonic wave described by

$$\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$

a) Determine an expression for the time average of $\langle S \rangle$. starting at t_0 and ending at t_0+T_i

b) Evaluate this for an average over one cycle starting to t_0 and ending at $t_0 + T$ where T is the period of the wave.

c) Repeat this over the interval $t_0 \to t_0+T$ where $T \gg 1$.

Answer: a)

$$\langle S \rangle = \epsilon_0 c E_0^2 \langle \cos^2(\vec{k}\cdot\vec{r} - \omega t) \rangle$$

and

$$\langle \cos^2(\ldots) \rangle = \frac{1}{T} \int_{t_0}^{T+t_0} \cos^2(\vec{k}\cdot\vec{r} - \omega t)\, dt.$$

$2\cos^2\theta - 1 = \cos 2\theta$

$$= \frac{1}{2T} \int_{t_0}^{T+t_0} \left[\cos(2\vec{k}\cdot\vec{r} - 2\omega t) + 1\right] dt.$$

$$= \frac{1}{2T} \left\{ \frac{\sin(2\vec{k}\cdot\vec{r} - 2\omega t)}{-2\omega} \Big|_{t_0}^{T+t_0} + T \right\}.$$

$$= \frac{1}{2} - \frac{1}{4\omega T} \left[\sin(2\vec{k}\cdot\vec{r} - 2\omega(T+t_0)) - \sin(2\vec{k}\cdot\vec{r} - 2\omega t_0)\right]$$

b) over one period $\sin(2\vec{k}\cdot\vec{r} - 2\omega(T+t_0)) = \sin(2\vec{k}\cdot\vec{r} - 2\omega t_0)$

$$\Rightarrow \langle \cos^2(\ldots) \rangle = \frac{1}{2} \quad \Rightarrow \quad \langle S \rangle = \frac{1}{2}\epsilon_0 c E_0^2$$

c) Here $|\sin(\ldots) - \sin(\ldots)| \leq 2$ and so as $T \to \infty$ the latter term $\to 0$ giving

$$\langle \cos^2(\ldots) \rangle = 1/2 \quad \Rightarrow \quad \langle S \rangle = \frac{1}{2}\epsilon_0 c E_0^2$$

Thus we aim to compute the time average of the Poynting vector

$$\langle S \rangle = \epsilon_0 c E_0^2 \langle \cos^2(\vec{k}\cdot\vec{r} - \omega t) \rangle.$$

In optics the time average of the Poynting vector is called the <u>irradiance</u>.

$$I := \langle S \rangle$$

and for a monochromatic wave (i.e. a harmonic wave)

$$I = \frac{1}{2} \epsilon_0 c E_0^2$$

Thus

$$\boxed{\text{Irradiance or Intensity is proportional to the square of the magnitude of the electric field.}}$$