HW 5 due Monday

Magnetic Fields in Electromagnetic Waves

Recall that electromagnetic fields in a vacuum satisfy

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

These can be combined to yield wave equations:

$$\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

Consider harmonic plane wave solutions, which, for the electric field have the form

$$\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t+\varphi)}$$

This has the properties:

i) the angular frequency + wavenumber are related by:

$$c = \omega/k \qquad \text{where} \qquad c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

is the speed with which these waves propagate.

ii) the direction along which waves propagate is $\vec{k}$

iii) the equation $\vec{\nabla}\cdot\vec{E}$ implies that $\vec{k}\cdot\vec{E}=0$. Thus

$$\boxed{\text{At every location, the electric field is perpendicular to the direction of propagation.}}$$

This is thus a transverse wave.

iv) If $\vec{E}_0$ is independent of $\vec{r}$ and t then at every location the electric field lies along the same line. At any single location it oscillates back + forth along one line. It is the called linearly polarized and the direction of polarization is determined by $\vec{E}_0$.

Demo: PSU Schuylkill demos

The magnetic field satisfies a similar equation and we might expect that harmonic wave solutions exist. This is true. However, it is important to note that the magnetic field depends on the electric field since

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Exercise 1 Consider an electromagnetic wave that has the following properties:

i) It propagates along the $\hat{z}$ direction with wavelength 3.0m

ii) It is linearly polarized along the x direction, and the electric field has amplitude 20 N/C.

The purpose of this exercise is to show that these completely specify the electric and magnetic fields for the wave.

a) The electric field has form

$$\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Determine expressions for $\vec{k}$, ω and $\vec{E}_0$

b) Using $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, determine an expression for the magnetic field. Ensure that you find expression for its frequency, wave vector, and the amplitude (a vector).

c) Show that

$$\omega \vec{B} = \vec{k} \times \vec{E}$$

Answer: i) First $\vec{k} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$ gives the direction of propagation. Thus $k_x = k_y = 0$ and

$$\vec{k} = k\hat{k}$$ unit vector along $\hat{z}$

Now $k = 2\pi/\lambda = \frac{2\pi}{3.0}\,m^{-1} = 2.1\,m^{-1}$

So $\boxed{\vec{k} = 2.1\,m^{-1}\,\hat{k}}$

Then $v = \omega/k \Rightarrow \omega = vk$

$$= 3.0\times10^8\,m/s \times \frac{2\pi}{3.0}\,m^{-1}$$

$$= 2\pi\times10^8/s = 6.3\times10^8\,Hz$$

$\boxed{\omega = 6.3\times10^8\,Hz.}$

Finally $\vec{E}_o = E_o\,\hat{i} = 20N/c\;\hat{i} \Rightarrow \boxed{\vec{E}_o = 20N/c\;\hat{i}}$

These completely specify $\vec{E}$.

ii) $\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(\vec{E}_o\, e^{i(k_z z - \omega t)}\right)$

$$= \vec{\nabla}\times\left[\vec{E}_o\, e^{i(kz-\omega t)}\right]$$

Using $\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times\vec{\nabla}f$ with $\vec{A} = \vec{E}_o$ gives:

$$\vec{\nabla}\times\vec{E} = -\vec{E}_o\times\vec{\nabla}\left(e^{i(kz-\omega t)}\right)$$

$$= -\vec{E}_o\times\left[\frac{\partial}{\partial x}(\ldots)\hat{i} + \ldots + \frac{\partial}{\partial z}(\ldots)\hat{k}\right]$$

$$= -\vec{E}_o\times ik\,e^{i(kz-\omega t)}\,\hat{k}$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{E} = -ik\left[E_0 \underbrace{\hat{i} \times \hat{k}}_{-\hat{j}}\right] e^{i(kz-\omega t)}$$

$$= ik\, E_0 e^{i(kz-\omega t)}\,\hat{j} = -\frac{\partial \vec{B}}{\partial t}$$

Integrating gives:

$$\vec{B} = \frac{k}{\omega} E_0\, e^{i(kz-\omega t)}\,\hat{j}$$

We see that since we know k, ω, E_0 we can completely specify $\vec{B}$, once $\vec{E}$ has been specified. Note that

$$k/\omega = 1/c$$

means

$$\vec{B} = \frac{E_0}{c} e^{i(kz-\omega t)}\,\hat{j}$$

and the amplitude of this is: $E_0/c = \dfrac{20\,N/C}{3.0 \times 10^8\,m/s} = 6.7 \times 10^{-8}\,T$

c) $$\vec{k} \times \vec{E} = k\hat{k} \times E_0 \hat{i} e^{i(kz-\omega t)}$$

$$= kE_0 \hat{j} e^{i(kz-\omega t)}$$

$$= \frac{\omega}{c} E_0\, e^{i(kz-\omega t}\,\hat{j}$$

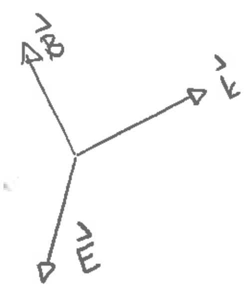
$$= \omega \vec{B}$$

This illustrates the fact that

> Specifying the electric field completely, results in a complete specification of the magnetic field via
>
> $$\omega \vec{B} = \vec{k} \times \vec{E}$$

We see that the

> Direction of propagation, the electric field and the magnetic field are all perpendicular.

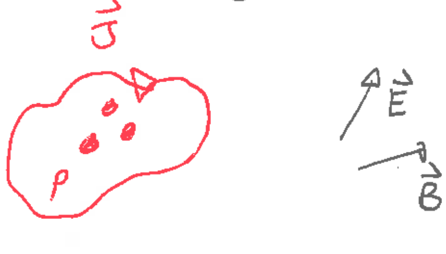


These facts are true for any electromagnetic fields of the form:

$$\vec{E} = \vec{E}_0 f(\vec{k}\cdot\vec{r} - \omega t)$$

Energy + electromagnetic waves

The energy required to assemble a collection of source charges + source currents can be computed by applying Newtonian mechanics to classical electromagnetism. The resulting energy can be determined from the fields produced by these charges. The result is that:



sources

The energy required to assemble these is:

$$\mathcal{U} = \frac{1}{2}\int_{\text{all space}} \left[\epsilon_0 \vec{E}\cdot\vec{E} + \frac{1}{\mu_0}\vec{B}\cdot\vec{B}\right] d\tau$$

We then broaden this to state:

> The energy contain within any region of space is:
>
> $$\mathcal{U} = \frac{1}{2}\int_{\text{region}} \left[\epsilon_0 \vec{E}\cdot\vec{E} + \frac{1}{\mu_0}\vec{B}\cdot\vec{B}\right] d\tau.$$
>
> and the energy per unit volume is
>
> $$u = \frac{\epsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}$$

With waves it should be clear that energy can be transported. Specifically with transverse electromagnetic waves we know that

$$\omega B = kE \Rightarrow B = \frac{k}{\omega}E = E/c.$$

for the magnitudes.

Thus the energy density is:

$$u = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} \frac{E^2}{c^2}$$

But $c^2 = \frac{1}{\mu_0 \epsilon_0}$ implies:

For transverse electromagnetic waves the energy density is:

$$\boxed{u = \epsilon_0 E^2 = \frac{B^2}{\mu_0}}$$

Again we only need electric field information to describe energy content.

Exercise 2: a) Consider the electromagnetic wave described by

$$\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$

where $\vec{k} = k\hat{i}$ and $\vec{E}_0 = E_0\hat{j}$.

Determine an expression for the energy density. Show that this depends on position and time.

b) Suppose

$$\vec{E} = \vec{E}_0 \, e^{-u^2/2}$$

where $u = \vec{k}\cdot\vec{r} - \omega t$ and $\vec{k} = k\hat{i}$, $\vec{E}_0 = E_0\hat{j}$.

Sketch the energy density as a function of x at t=0s

How does it appear later?

Answers: a) $E^2 = \vec{E} \cdot \vec{E}$

$= E_0^2 \cos^2(kx - \omega t)$

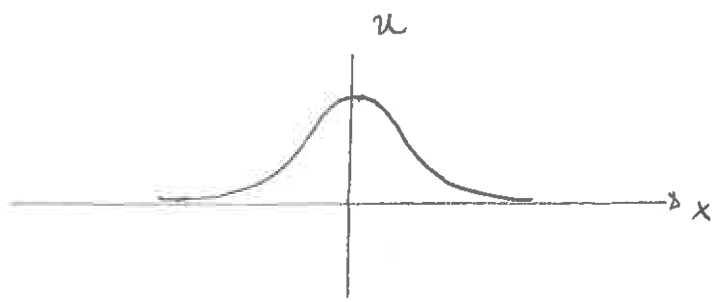
$\Rightarrow \quad u = \epsilon_0 E_0^2 \cos^2(kx - \omega t)$

depends on position + time

b) $E^2 = E_0^2 e^{-u^2}$

$= E_0^2 e^{-(kx - \omega t)^2}$

At $t = 0$



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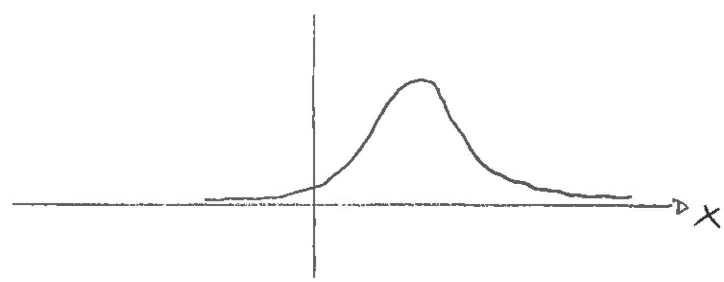

Energy is clearly being transported.