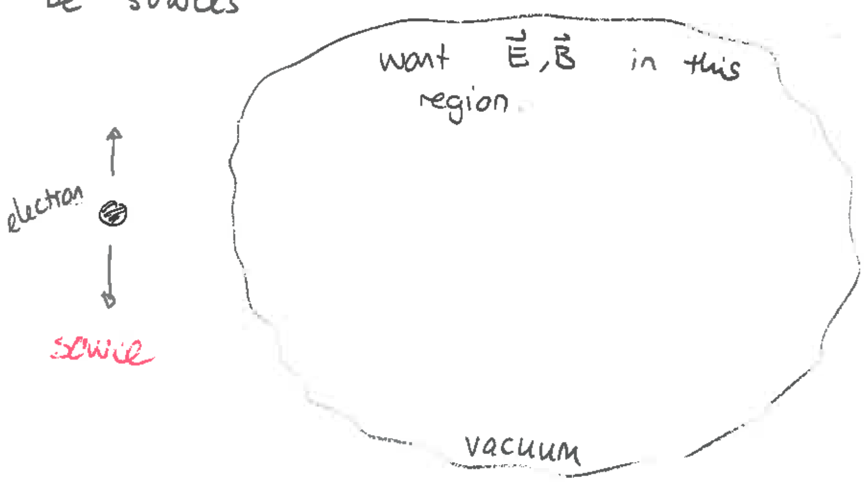
Electromagnetic Fields in a Vacuum

Maxwell's equations in the general case are:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}.$$

In optics we will consider the fields in a region where there are no source charges or currents although outside this region there may be sources



In this vacuum region $\rho = 0$ and $\vec{J} = 0$. So Maxwell's equations become:

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

What fields (aside from $\vec{E} = \vec{B} = 0$) could satisfy these?

Exercise 1: A vector calculus rule is:

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{A}] = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

Use this to obtain a differential equation that only involves $\vec{E}$ and not $\vec{B}$. Also obtain a differential equation that only involves $\vec{B}$ and not $\vec{E}$. (in a vacuum)

Answer:

$$\vec{\nabla} \times \underbrace{(\vec{\nabla} \times \vec{E})}_{-\frac{\partial \vec{B}}{\partial t}} = \vec{\nabla}\underbrace{(\vec{\nabla} \cdot \vec{E})}_{0} - \nabla^2 \vec{E}$$

Maxwell eqn.

$$\Rightarrow \quad \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\nabla^2 \vec{E} \quad \Rightarrow \quad -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\nabla^2 \vec{E}$$

and $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ gives:

$$-\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = -\nabla^2 \vec{E}$$

$$\Rightarrow \quad \nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Similarly

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}\underbrace{(\vec{\nabla} \cdot \vec{B})}_{0} - \nabla^2 \vec{B}$$

$$\Rightarrow \quad \vec{\nabla} \times \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = -\nabla^2 \vec{B}$$

$$\Rightarrow \quad \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{\nabla} \times \vec{E} = \ldots$$

$$\Rightarrow \quad \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right) = -\nabla^2 \vec{B}$$

$$\Rightarrow \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

We see that:

In a vacuum the electric + magnetic fields satisfy

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Each of these is a wave equation for the three field components. For example,

$$\nabla^2 E_x = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

$$\nabla^2 E_y = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

Thus Maxwell's equations predict the existence of electromagnetic waves in a vacuum.

Exercise Using $\epsilon_0 = 8.85 \times 10^{-12}\ C^2/Nm^2$ and $\mu_0 = 4\pi \times 10^{-7}\ Tm/A$, determine the wave speed of electromagnetic waves in a vacuum.

Answer: $\frac{1}{v^2} = \mu_0 \epsilon_0 \Rightarrow \quad v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

$$= \frac{1}{\sqrt{8.85 \times 10^{-12}\ C^2/N\cancel{m} \cdot 4\pi \times 10^{-7}\ T\cancel{m}/A}}$$

$$= 2.999 \times 10^8\ m/s$$

Thus

$$\boxed{\text{In a vacuum electromagnetic waves propagate at the speed of light, } c}$$

Thus $c = 1/\sqrt{\mu_0 \epsilon_0}$

In a dielectric medium, which is one that produces a local electric field in response to an external field, the permittivity of free space ϵ_0 is replaced by the permittivity ϵ in Maxwell's equations. We find the wavespeed is

$$V = \frac{1}{\sqrt{\epsilon \mu_0}} \neq c$$

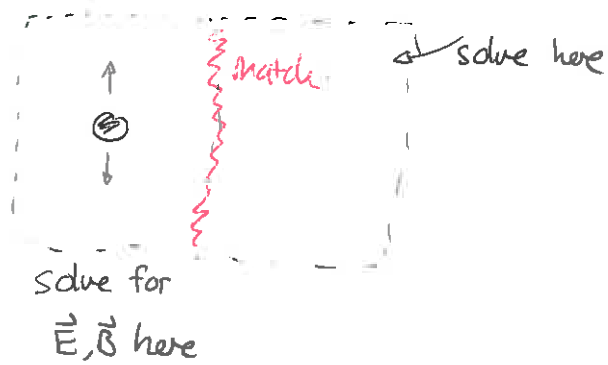
The index of refraction of the material is then.

$$n = c/V$$

Harmonic plane wave solutions

The solutions to the wave equation in the region will depend on the nature of the sources beyond the region. The reason is that one also has to solve Maxwell's equations in the region containing the sources and then match solutions on the boundary.

We will ignore this issue for a moment and consider a general type of solution, the harmonic plane wave solution. For the electric field the complex representation of the harmonic plane wave solution is:



$$\boxed{\vec{E}(\vec{r},t) = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t + \varphi)}}$$

independent of $\vec{r}, t$.

As before this represents a wave with plane crests and which travels in the direction of $\vec{k}$ with speed

$$c = \omega/k.$$

However every point along the wave is no longer described by a single real displacement variable but rather by a vector. The real solution is:

$$\vec{E}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \varphi)$$

and we see a vector amplitude $\vec{E}_0$ rather than a scalar amplitude. We first consider the direction of this vector and then the magnetic field

Exercise 3

a) The electric field must satisfy

$$\vec{\nabla}\cdot\vec{E}=0$$

in a vacuum. Show that this implies:

$$\vec{k}\cdot\vec{E}=0$$

b) Suppose $\vec{k}=k\hat{i}$ and $\vec{E}_0=E_0\hat{j}$ and $\varphi=0$. Sketch a snapshot of the wave (along x) at $t=0$.

c) Suppose $\vec{k}=k\hat{i}$ and $\vec{E}_0=E_0\hat{j}$. Consider the field at $x=0$. As time passes what does it do?

Answers: a) $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t+\varphi)}\right]$

$$= e^{i(\vec{k}\cdot\vec{r}-\omega t+\varphi)} \underbrace{\vec{\nabla}\cdot\vec{E}_0}_{=0} + \vec{E}_0\cdot\vec{\nabla}\left[e^{i(\vec{k}\cdot\vec{r}-\omega t+\varphi)}\right]$$

Then $\vec{\nabla} e^{i(\vec{k}\cdot\vec{r}-\dots)} = \frac{\partial}{\partial x} e^{i(\vec{k}\cdot\vec{r}-\omega t+\varphi)}\hat{i} + \frac{\partial}{\partial y} e^{i(\vec{k}\cdot\vec{r}-\omega t+\varphi)}\hat{j} + \dots$

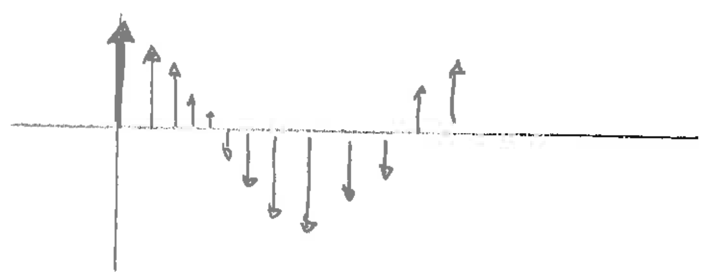
$$= \left[k_x\hat{i}+k_y\hat{j}+k_z\hat{k}\right] e^{i(\dots)}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \vec{E}_0 e^{i(\dots)}\cdot\vec{k} = \vec{k}\cdot\vec{E}$$

$$\Rightarrow \vec{k}\cdot\vec{E} = 0$$

b) $\vec{E}(\vec{r},t) = \vec{E}_0\cos(\vec{k}\cdot\vec{r}-\omega t+\varphi)$

$$= E_0\hat{j}\cos(kx)$$



c) $\vec{E}(\vec{r},t) = E_0\hat{j}\cos(\omega t)$ oscillates up+down along y axis.

Thus

$\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t + \varphi)}$ describes a wave, whose electric field is perpendicular to direction of propagation. At any single point the field oscillates up and down along a line in the direction of $\vec{E}_0$. This is the same at all points.

The fact that it oscillates along one direction is described by saying that the wave is transverse.

The fact that it oscillates along the same direction at all points is described by saying that it is linearly polarized