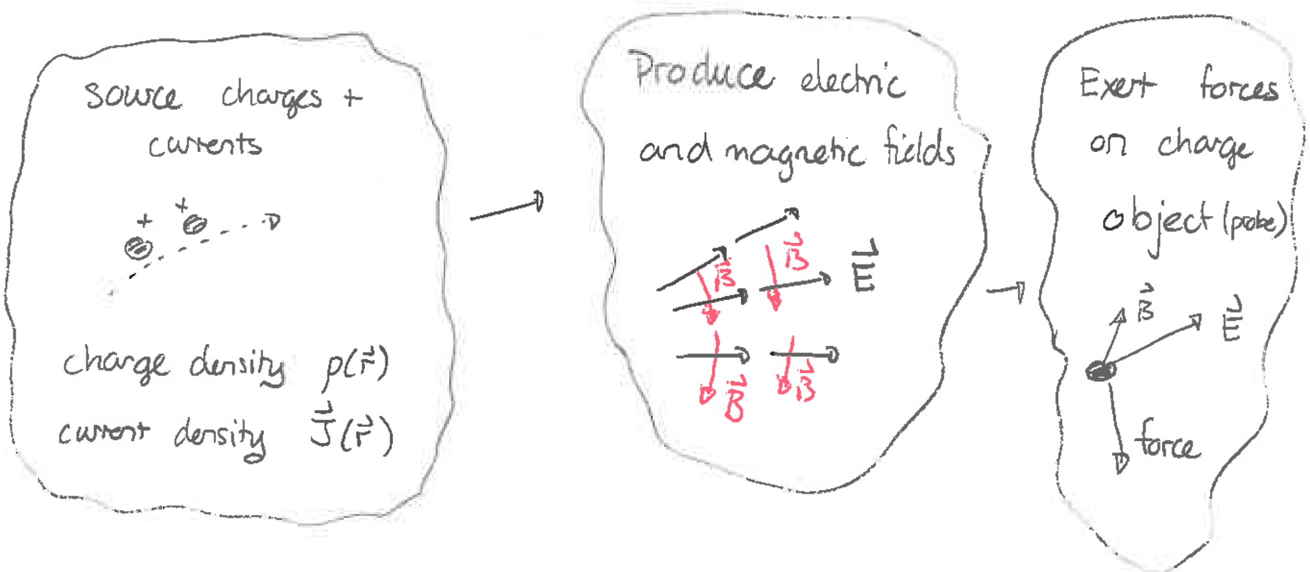
HW 3 today

HW 4 Friday

Electromagnetism

Electromagnetism describes how charged objects interact. The basic structure is illustrated below:



The Lorentz force law gives:

> If a probe charge with charge q is at a location where the electric field is $\vec{E}$ and the magnetic field is $\vec{B}$ then the electromagnetic force on the particle is:
>
> $$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

The rules for production of these fields involve sources described by a charge density $\rho(\vec{r})$ and current density $\vec{J}(\vec{r})$. The meaning of these is given via:

$$\boxed{\text{Charge in region} = Q = \int_{\text{region}} \rho(\vec{r})\, d\tau}$$

and

$$\text{Current flowing across surface} = I = \int_{\text{Surface}} \vec{J}(\vec{r}) \cdot d\vec{a}$$

The electric + magnetic fields satisfy a set of differential equations called Maxwell's equations and these are:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho(\vec{r})$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Where ϵ_0 is the permittivity of free space and μ_0 is the permeability of free space. In various materials these are replaced by their non-free space variants ϵ, μ.

These can be converted into integral forms using the divergence theorem

$$\int_{\text{enclosed region}} \vec{\nabla} \cdot \vec{A} \, d\tau = \oint_{\text{surface}} \vec{A} \cdot d\vec{a}$$

and Stoke's theorem:

$$\int_{\text{surface}} \vec{\nabla} \times \vec{A} \cdot d\vec{a} = \oint_{\text{bounding loop.}} \vec{A} \cdot d\vec{l}$$

Electrostatic Fields.

These can be used to compute electric fields for static charge arrangements where $\vec{J}(\vec{r}) = 0$. In this case $\vec{E}$ is independent of t and thus $\vec{B} = 0$. So we require

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho(\vec{r})$$

$$\vec{\nabla} \times \vec{E} = 0$$

Exercise 1: Consider a sphere of radius a with charge density

$$\rho(\vec{r}) = \alpha r^2$$

a) Determine the total charge inside the sphere Q and use this to rewrite α in terms of Q.

b) Determine the electric field at all locations.

Answer: a) $$Q = \int_{\text{sphere}} \rho \, d\tau \qquad d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$0 \leq r \leq a$$
$$0 \leq \theta \leq \pi$$
$$0 \leq \phi \leq 2\pi$$

$$= \alpha \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{a} dr \, r^2 \sin\theta \, \alpha r^2$$

$$= \alpha 2\pi \int_0^{\pi} \sin\theta \, d\theta \int_0^{a} r^4 dr$$

$$= \frac{4\pi a^5}{5} \alpha \Rightarrow \alpha = \frac{5}{4\pi a^5} Q$$

b) By symmetry $\vec{E} = E_r(r)\,\hat{r}$. Now Stoke's Law gives:

$$\underbrace{\int}_{\substack{\text{enclosed}\\ \text{region}}} \vec{\nabla}\cdot\vec{E}\,d\tau = \underbrace{\oint}_{\text{surface}} \vec{E}\cdot d\vec{a} \quad \Rightarrow \quad \underbrace{\int \frac{1}{\epsilon_0}\rho\,d\tau}_{\frac{1}{\epsilon_0} q_{enc}} = \oint \vec{E}\cdot d\vec{a}$$

Choose as a surface a sphere of radius r. Then

$$d\vec{a} = r^2 \sin\theta\, d\theta\, d\phi \qquad 0 \le \theta \le \pi$$
$$0 \le \phi \le 2\pi$$

and

$$\vec{E}\cdot d\vec{a} = E_r(r)\, r^2 \sin\theta\, d\theta\, d\phi$$

and so

$$\frac{1}{\epsilon_0}\int \rho\, d\tau = \int_0^{2\pi}\int_0^{\pi} E_r(r)\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= E_r(r)\, r^2\, 4\pi$$

$$\Rightarrow \quad E_r(r) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int_{\text{region}} \rho\, d\tau$$

If $r > a$ then $\int \rho\, d\tau = Q \Rightarrow E_r(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \Rightarrow \overset{r>a}{\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}}}$

If $r < a$ then $\int \rho\, d\tau = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{r} dr\, r^2 \sin\theta\, \rho(r)$

$$= 4\pi \int_0^r \alpha r^4\, dr = \frac{4\pi}{5}\alpha r^5 = Q\left(\frac{r}{a}\right)^5$$

So $\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\left(\frac{r}{a}\right)^5\hat{r}}$

Time-varying electric + magnetic fields

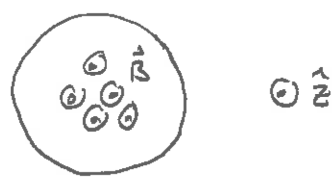
We will consider fields in regions where there are no source charges or currents. In these regions Maxwell's equations become:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Exercise 2 A cylindrical region of radius R contains a uniform magnetic field $\vec{B} = B_0 e^{-t/\tau} \hat{z}$ where τ is a constant with units of time. Determine the electric field that this induces.

Answer:



In order to produce a field whose curl is along $\hat{z}$ we must have $\vec{E}$ in a direction along $\hat{\phi}$. So $\vec{E} = E_\phi(r)\,\hat{\phi}$.

Now Stokes Law and $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ gives:

$$\int_{\text{loop surface}} \vec{\nabla} \times \vec{E} \cdot d\vec{a} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{a}$$

$$\Rightarrow \oint_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{a}$$

We take as a loop a circle radius r centered on the z axis. Then

$$d\vec{l} = r\, d\phi\, \hat{\phi}$$

$$\Rightarrow \vec{E} \cdot d\vec{l} = r\, d\phi\, \hat{\phi} \cdot E_\phi \hat{\phi} = r E_\phi\, d\phi$$

Thus

$$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{a}$$

$$\Rightarrow \int_0^{2\pi} r\,E_\phi(r)\,d\phi = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{a}$$

$$\Rightarrow r\,E_\phi(r)\,2\pi = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{a}$$

Then if $r < R$

$$\int \vec{B}\cdot d\vec{a} = \int \vec{B}\cdot r\,dr\,d\phi\,\hat{z}$$

$$= B_0 e^{-t/\tau}\int_0^r r'dr' \int_0^{2\pi} d\phi$$

$$= \pi r^2 B_0 e^{-t/\tau}.$$

if $r > R$ the integral terminates at $r = R \Rightarrow \int \vec{B}\cdot d\vec{a} = \pi R^2 B_0 e^{-t/\tau}$

So if $r < R$

$$E_\phi = -\frac{1}{2\pi r}\frac{\partial}{\partial t}\pi r^2 B_0 e^{-t/\tau} = -\frac{r}{2}B_0\left(-\frac{1}{\tau}\right)e^{-t/\tau}$$

$$\Rightarrow \vec{E} = \frac{r B_0}{2\tau}e^{-t/\tau}\,\hat{\phi} \qquad r < R$$

if $r > R$

$$E_\phi = -\frac{1}{2\cancel{\pi} r}\cancel{\pi}R^2 B_0\frac{\partial}{\partial t}e^{-t/\tau} = \frac{R^2 B_0}{2r\tau}e^{-t/\tau}$$

$$\Rightarrow \vec{E} = \frac{R^2 B_0}{2r\tau}e^{-t/\tau}\,\hat{\phi} \qquad r > R$$

Note that

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(r\cancel{E_r}^{\,0}) + \frac{1}{r}\cancel{\frac{\partial E_\phi}{\partial\phi}}^{\,0} + \frac{\partial \cancel{E_z}^{\,0}}{\partial z}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = 0 \quad \text{everywhere}$$

By construction $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$. Finally

$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$= \begin{cases} -\dfrac{\mu_0\epsilon_0}{\tau}\dfrac{rB_0}{2\tau}e^{-t/\tau}\,\hat{\phi} \\ \dfrac{\mu_0\epsilon_0}{\tau}\dfrac{R^2B_0}{2r\tau}e^{-t/\tau}\,\hat{\phi} \end{cases}$$

Then $\vec{\nabla}\times\vec{B} = \left[\frac{1}{r}\frac{\partial B_z}{\partial\phi} - \frac{\partial B_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\phi) - \frac{\partial B_r}{\partial\phi}\right]\hat{z}$

further constains $\vec{B}$. The given magnetic field would produce

$$\vec{\nabla}\times\vec{B} = 0$$

and this is clearly not consistent with the resulting electric field. There must be another magnetic field participating here.