Lecture 3

* HW 1 solutions posted to K drive
* HW 2 due later
* HW3 due Monday

Waves in Three Dimensions

Recall that the one dimensional wave equation is

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}$$

We would like to extend this to describe waves in more dimensions. Examples of higher dimensional wave phenomena are:

1) vibrations of two dimensional surfaces
2) vibrations of three dimensional objects
3) vibrations of air, usually described in terms of pressures.

Consider sound, which is described in terms of pressure. If the background pressure in the absence of any sound is P_0 and this is the same at all locations, then the pressure as a result of the presence of sound depends on location and time as:

$$\boxed{\text{Pressure at location } \vec{r} \text{ at time } t = P(\vec{r},t) = P_0 + P_{excess}(\vec{r},t)}$$

where $\vec{r}$ describes the location of various points at which pressure might be measured. The study of sound waves usually considers the excess pressure $P_{excess}(\vec{r},t)$

This is an example of a function $\Psi(\vec{r},t)$ which varies with location $\vec{r}$ and time t. In general we can express the location as

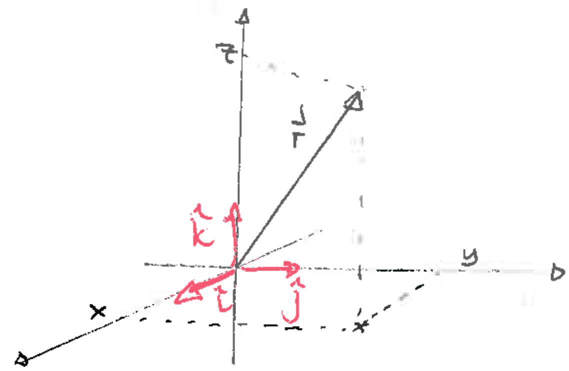
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

using the usual Cartesian co-ordinates

We can then use:

Disturbance described by

$$\Psi(\vec{r},t) = \Psi(x,y,z,t)$$



The equation that this satisfies depends on the underlying physics. However, in many physical situations we find that it satisfies the three dimensional wave equation:

$$\boxed{\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}}$$

An alternative notation for this uses the Laplacian operator:

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

and gives:

$$\boxed{\nabla^2 \Psi = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}}$$

We now explore solutions to this

Cartesian co-ordinates

The solutions to the three dimensional wave equation can be presented using Cartesian co-ordinates. These are suitable for certain situations. In general one can show explicitly that

A possible solution to the three dimensional wave equation is:

$$\Psi(\vec{r},t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \phi)$$

where $\vec{k} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$, and ω and ϕ are independent of x, y, z and t, and

$$\omega = v k$$

where $k = |\vec{k}|$.

The quantity $\vec{k}$ is called the wave vector.

An alternative real form of this type of solution is

$$\Psi(\vec{r},t) = A\sin(\vec{k}\cdot\vec{r} - \omega t + \phi)$$

and more complicated solutions can be obtained via superpositions of these basic harmonic solutions.

The associated complex representation of these solutions is:

$$\tilde{\Psi}(\vec{r},t) = A e^{i(\vec{k}\cdot\vec{r} - \omega t + \phi)}$$

and we extract a real solution via

$$\Psi(\vec{r},t) = \mathrm{Re}\left[\tilde{\Psi}(\vec{r},t)\right]$$

Exercise 1 Suppose that

$$\vec{k} = \frac{k}{\sqrt{2}}(\hat{\imath} + \hat{\jmath})$$

a) Show that $\tilde{\Psi}(\vec{r},t) = Ae^{i(\vec{k}\cdot\vec{r} - \omega t)}$ satisfies the three dimensional wave equation.

b) We aim to determine regions which return the same value for Ψ at any given instant. Consider $t=0$. Determine an expression for locations that yield maxima (wave crests) at this instant. What geometrical shape do these have? How are they related to the direction of $\vec{k}$?

c) As time passes, in which direction do these crests move?

Answers: a) $\vec{k}\cdot\vec{r} = \frac{k}{\sqrt{2}}(x+y)$. Thus

$$\tilde{\Psi}(\vec{r},t) = Ae^{i(kx/\sqrt{2} + ky/\sqrt{2} - \omega t)}$$

Then $$\frac{\partial\tilde{\Psi}}{\partial x} = \frac{ik}{\sqrt{2}} Ae^{i(\ldots)} = \frac{ik}{\sqrt{2}}\tilde{\Psi}$$

$$\frac{\partial^2\tilde{\Psi}}{\partial x^2} = \left(\frac{ik}{\sqrt{2}}\right)^2 \Psi = -\frac{k^2}{2}\tilde{\Psi}$$

Similarly

$$\frac{\partial^2\tilde{\Psi}}{\partial y^2} = \left(\frac{ik}{\sqrt{2}}\right)\tilde{\Psi} = -\frac{k^2}{2}\tilde{\Psi}$$

$$\frac{\partial^2\tilde{\Psi}}{\partial t^2} = -\omega^2\tilde{\Psi}$$

Substituting into the wave equation gives:

$$-\frac{k^2}{2}\tilde{\Psi} - \frac{k^2}{2}\tilde{\Psi} = -\frac{\omega^2}{v^2}\tilde{\Psi} \Leftrightarrow k^2\tilde{\Psi} = \frac{\omega^2}{v^2}\tilde{\Psi}$$ and this is satisfied if $\omega = kv$.

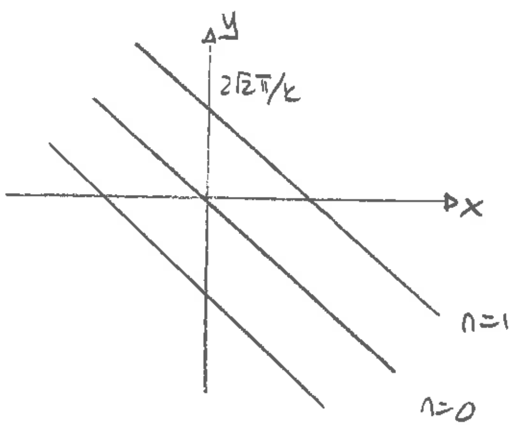
b) At $t=0$ $\quad \tilde{\Psi}(\vec{r},0) = Ae^{i(kx/\sqrt{2}+ky/\sqrt{2})} \Rightarrow \Psi(r,0) = A\cos(\dots\dots)$

We need $(kx+ky)/\sqrt{2} = 0, 2\pi, 4\pi, \dots = 2n\pi$. for any value of n

$$\frac{k}{\sqrt{2}}(x+y) = 2n\pi \Rightarrow x+y = \frac{2n\pi\sqrt{2}}{k}$$

$$\Rightarrow y = -x + \frac{2n\pi\sqrt{2}}{k}$$

These give straight lines with slope $= -1$

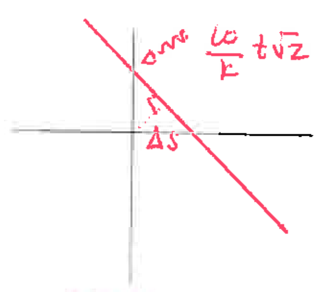


c) Consider the $n=0$ crest (at $t=0$). As time passes this follows the trajectory described by

$$\frac{k}{\sqrt{2}}(x+y) - \omega t = 0 \Rightarrow x+y = \frac{\omega t\sqrt{2}}{k}.$$

$$\Rightarrow y = -x + \frac{\omega t\sqrt{2}}{k}.$$

The slope of this line is still -1 but the y intercept increases



Distance traveled

$$\Delta s = \frac{1}{\cancel{\sqrt{2}}} \cdot \frac{\omega}{k} t \cancel{\sqrt{2}}$$

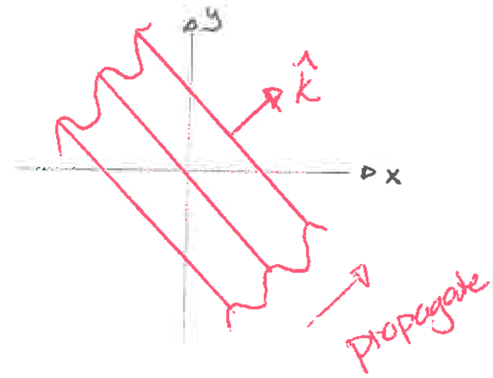
$$\Rightarrow \Delta s = \frac{\omega}{k} t$$

speed v

This illustrates the following:

> The solutions associated with $\tilde{\Psi} = Ae^{i(\vec{k}\cdot\vec{r} - \omega t + \phi)}$ have crests that form planes perpendicular to $\vec{k}$. These planes propagate with speed $v = \omega/k$ in the direction of $\vec{k}$.

Such solutions are called <u>plane waves</u>:



Spherical co-ordinates

When using spherical co-ordinates, the Laplacian becomes:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

and thus the wave equation takes the form:

$$\nabla^2\Psi = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

$$\Rightarrow \boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}}$$

While the general form of a solution to this might be very complicated, one important class of solutions are waves which are spherically symmetric. A spherically symmetric wave is one which is independent of θ, ϕ. Thus we seek $\Psi(\vec{r}, t) = \Psi(r,t)$. For spherically symmetric waves,

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}}$$

We can convert this into a complex representation. Then a special class of solutions are harmonic spherical waves:

$$\tilde{\Psi} = \frac{A}{r}e^{i(kr \mp \omega t + \varphi)}$$

Since

$$\frac{\partial\tilde{\Psi}}{\partial r} = -\frac{1}{r}\tilde{\Psi} + ik\tilde{\Psi} = \left(-\frac{1}{r} + ik\right)\tilde{\Psi}$$

$$\Rightarrow \quad r^2\frac{\partial\tilde{\Psi}}{\partial r} = -r\tilde{\Psi} + ikr^2\tilde{\Psi}$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial \tilde{\Psi}}{\partial r}\right) = -\tilde{\Psi} - r\frac{\partial \tilde{\Psi}}{\partial r} + 2ikr\,\tilde{\Psi} + ikr^2 \frac{\partial \tilde{\Psi}}{\partial r}$$

$$= \tilde{\Psi}\left(-1+2ikr\right) + \left(-r+ikr^2\right)\frac{\partial \tilde{\Psi}}{\partial r}$$

$$= \tilde{\Psi}\left(-1+2ikr\right) + \left(-r+ikr^2\right)\left(-\frac{1}{r}+ik\right)\tilde{\Psi}$$

$$= \tilde{\Psi}\left[\cancel{-1}\cancel{+2ikr}\ \cancel{+1} - \cancel{ikr} \cancel{-ikr} - k^2 r^2\right]$$

$$= -k^2 r^2 \tilde{\Psi}$$

So $\quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \tilde{\Psi}}{\partial r}\right) = -k^2\tilde{\Psi}$

Then $\quad \frac{\partial^2 \tilde{\Psi}}{\partial t^2} = -\omega^2\tilde{\Psi}$ gives

$$-k^2\tilde{\Psi} = -\frac{\omega^2}{v^2}\tilde{\Psi}$$

which is again a solution if $\omega = kv$.

The associated real harmonic spherical wave is

$$\boxed{\Psi(r,t) = \frac{A}{r}\cos\left(kr \mp \omega t + \varphi\right)}$$

The wave attains maxima when $r =$ constant, i.e. on spherical shells centered at the origin. These travel outward for the $-$ sign and inward for the plus sign.