Phys 473 2015

Lecture 2

HW 2 due Friday

Superpositions of waves

A characteristic feature of waves is the fact that two or more waves can combine to produce a distinct new wave. This is called the principle of superposition and the resulting combination is called a superposition. Mathematically;

> If Ψ_1 and Ψ_2 each solve the wave equation and α_1 and α_2 are any constants then
>
> $$\Psi = \alpha_1 \Psi_1 + \alpha_2 \Psi_2$$
>
> also satisfies the wave equation.

Proof: We need to show that $\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 \Psi}{\partial t^2}$. To do this:

$$\frac{\partial \Psi}{\partial x} = \alpha_1 \frac{\partial \Psi_1}{\partial x} + \alpha_2 \frac{\partial \Psi_2}{\partial x} \Rightarrow \frac{\partial^2 \Psi}{\partial x^2} = \alpha_1 \frac{\partial^2 \Psi_1}{\partial x^2} + \alpha_2 \frac{\partial^2 \Psi_2}{\partial x^2}$$

But Ψ_1 solves the wave equation. Thus $\frac{\partial^2 \Psi_1}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 \Psi_1}{\partial t^2}$ and $\frac{\partial^2 \Psi_2}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 \Psi_1}{\partial t^2}$. Thus

$$\frac{\partial^2 \Psi}{\partial x^2} = \alpha_1 \frac{1}{V^2}\frac{\partial^2 \Psi_1}{\partial t^2} + \alpha_2 \frac{1}{V^2}\frac{\partial^2 \Psi_2}{\partial t^2} = \frac{1}{V^2}\left[\alpha_1 \frac{\partial^2 \Psi_1}{\partial t^2} + \alpha_2 \frac{\partial^2 \Psi_2}{\partial t^2}\right]$$

By a similar reasoning to that above, $\frac{\partial^2 \Psi}{\partial t^2} = \alpha_1 \frac{\partial^2 \Psi_1}{\partial t^2} + \alpha_2 \frac{\partial^2 \Psi_2}{\partial t^2}$

Thus $\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 \Psi}{\partial t^2}$ $\blacksquare$

<u>Exercise</u>: Let

$$\Psi_1(x,t) = A\sin(kx - \omega t)$$

$$\Psi_2(x,t) = A\sin(kx + \omega t).$$

a) In which direction does the wave represented by $\Psi_1(x,t)$ travel? What about $\Psi_2(x,t)$?

b) Let

$$\Psi(x,t) = \Psi_1(x,t) + \Psi_2(x,t)$$

Simplify this and show by direct substitution into the wave equation that it satisfies the wave equation. Show that the wave neither travels left nor right.

<u>Answer</u>: a) $\Psi_1(x,t) = A\sin\left[k(x - vt)\right]$ with $v = \omega/k$ implies a right-moving wave

$\Psi_2(x,t) = A\sin\left[k(x + vt)\right]$ implies a left moving wave

b) $$\Psi(x,t) = A\sin(kx - \omega t) + A\sin(kx + \omega t)$$

$$= A\left[\sin kx\cos\omega t - \sin\omega t\cos kx\right]$$

$$+ A\left[\sin kx\cos\omega t + \sin\omega t\cos kx\right]$$

$$= 2A\sin kx\cos\omega t$$

This has peaks (maxima) at $x = \pm\frac{\pi}{2k}, \pm\frac{3\pi}{2k}$ ...... etc regardless of time. If the peaks are at fixed locations, then the wave is not traveling.

<u>Demo</u>: ZonaLand Wave Interference 2 - sinusoidal # 6 or sinusoidal 7

If any physical entity is described via a wave equation, then mathematics asserts that superpositions of distinct waves must exist. We would therefore have to hunt for signs of such superpositions and we shall see that, in optics, this evidence comes in the form of interference and diffraction.

Complex numbers

A more convenient way of manipulating harmonic traveling waves uses complex numbers. Recall that a general complex number has form:

$$z = a + ib$$

where a, b are real and $i^2 = -1$. Complex numbers obey the usual addition and multiplication rules subject to $i^2 = -1$. Additional nomenclature and definitions are:

1) $\boxed{\text{If } z = a + ib \text{ then the real part of } z \text{ is } \operatorname{Re}[z] := a \quad \text{" imaginary " " " " } \operatorname{Im}[z] := b}$

2) the complex conjugate is defined by:

$$\boxed{\text{If } z = a + ib \quad \text{then} \quad z^* = a - ib}$$

3) the magnitude of a complex number is defined via:

$$\boxed{\text{If } z = a + ib \quad \text{then} \quad |z| = \sqrt{a^2 + b^2}}$$

We can show that

1) $\operatorname{Re}[z] = \dfrac{z + z^*}{2}$ $\qquad\qquad$ $\operatorname{Im}[z] = \dfrac{z - z^*}{2i}$

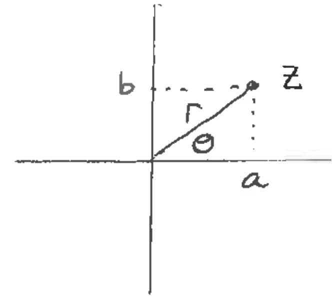
2) $(z_1 + z_2)^* = z_1^* + z_2^*$

$(z_1 z_2)^* = z_1^* z_2^*$

$z z^* = |z|^2$

3) $|z_1 z_2| = |z_1||z_2|$

It is convenient to represent complex numbers in the complex plane. An alternative representation uses polar co-ordinates r, θ. Note that $r = |z|$.



Then

$$z = r\cos\theta + i r\sin\theta$$

$$z = r(\cos\theta + i\sin\theta)$$

and this motivates the definition of the complex exponential:

$$\boxed{e^{i\theta} := \cos\theta + i\sin\theta} \quad \Rightarrow \quad z = re^{i\theta}$$

This is called the Euler relation. We can use trigonometric identities to show that

1) $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

2) $(e^{i\theta})^* = e^{-i\theta}$

3) $|e^{i\theta}| = 1$

4) $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

5) $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

Then in general $\quad z = re^{i\theta}$

We can see that a typical harmonic traveling wave can be represented as

$$\boxed{\Psi(x,t) = A\cos(kx \mp \omega t + \phi) = \mathrm{Re}\left[Ae^{i(kx \mp \omega t + \phi)}\right]}$$

It is very common to use the complex wave representation:

$$\tilde{\Psi}(x,t) = Ae^{i(kx \mp \omega t + \phi)}$$

where the real solution is $\quad \Psi(x,t) = \mathrm{Re}\left[\tilde{\Psi}(x,t)\right]$

The definition of the complex exponential can be used to show that

$$\boxed{\frac{d}{d\theta} e^{i\theta} = i e^{i\theta}}$$

<u>Exercise</u>: a) Show by direct substitution that $\tilde{\Psi}(x,t)$ satisfies the wave equation.

b) Consider the two solutions

$$\Psi_1(x,t) = A\cos(kx-\omega t)$$

$$\Psi_2(x,t) = A\cos(kx+\omega t)$$

We aim to add these to form $\Psi(x,t) = \Psi_1(x,t) + \Psi_2(x,t)$.

Do this by using the complex representation and show that the resulting expression has the form

$$\tilde{\Psi}(x,t) = (\text{function of } x) \times \text{function of } t.$$

Use $\Psi(x,t) = \mathrm{Re}\left[\tilde{\Psi}(x,t)\right]$ to obtain the real solution.

<u>Answers</u>: a) $\dfrac{\partial \tilde{\Psi}}{\partial x} = \dfrac{\partial}{\partial x} A e^{i(kx-\omega t+\phi)} = ik\,A e^{i(kx-\omega t+\phi)}$

$$\frac{\partial^2 \tilde{\Psi}}{\partial x^2} = ikA \frac{\partial}{\partial x} e^{i(kx-\omega t+\phi)} = ikA\, ik\, e^{i(kx-\omega t+\phi)}$$

$$\Rightarrow \frac{\partial^2 \tilde{\Psi}}{\partial x^2} = -k^2 A e^{i(kx-\omega t+\phi)}$$

Similarly $\dfrac{\partial^2 \tilde{\Psi}}{\partial t^2} = -\omega^2 A e^{i(kx-\omega t+\phi)}$

Substituting into

$$\frac{\partial^2 \tilde{\Psi}}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \tilde{\Psi}}{\partial t^2}$$

gives:

$$-k^2 \cancel{A e^{i(kx-\omega t+\phi)}} = \frac{1}{v^2}\left[-\omega^2 \cancel{A e^{i(kx-\omega t+\phi)}}\right]$$

$$\Rightarrow \quad k^2 = \frac{\omega^2}{v^2}$$

So if $k = \omega/v$ then this is a solution.

b) $\tilde{\Psi} = \tilde{\Psi}_1 + \tilde{\Psi}_2$

and $\tilde{\Psi}_1 = A e^{i(kx-\omega t)}$

$\tilde{\Psi}_2 = A e^{i(kx+\omega t)}$

So $\tilde{\Psi} = \tilde{\Psi}_1 + \tilde{\Psi}_2 = A e^{i(kx-\omega t)} + A e^{i(kx+\omega t)}$

$$= A e^{ikx} e^{-i\omega t} + A e^{ikx} e^{i\omega t}$$

$$= A e^{ikx}\left[e^{-i\omega t} + e^{i\omega t}\right]$$

$$\Rightarrow \quad \tilde{\Psi} = A e^{ikx}\, 2\cos(\omega t) = 2A\, e^{ikx} \cos(\omega t)$$

Then $\Psi(x,t) = Re\left[\tilde{\Psi}(x,t)\right] = Re\left[2A\, e^{ikx}\cos(\omega t)\right]$

$$= 2A\; Re\left[e^{ikx}\right]\cos(\omega t)$$

$$= 2A \cos(kx)\cos(\omega t)$$