Modern Optics: Homework 17

Due: 21 October 2015

1 Superposition of two waves of different amplitudes

Consider three waves, all produced at the same source with the same phase and polarization.

$$\begin{aligned}\tilde{\mathbf{E}}_1 &= E_1 e^{i(k_1 x - \omega_1 t)}\hat{\mathbf{y}} \\ \tilde{\mathbf{E}}_2 &= E_2 e^{i(k_2 x - \omega_2 t)}\hat{\mathbf{y}}\end{aligned}$$

a) Defining $\delta_j := k_j x - \omega_j t$, show that the superposition is

$$\tilde{\mathbf{E}} = e^{i(\delta_1+\delta_2)/2}\; Z\; \hat{\mathbf{y}}$$

where

$$Z := E_1 e^{i(\delta_1-\delta_2)/2} + E_2 e^{-i(\delta_1-\delta_2)/2}.$$

b) Then Z describes the envelope function since it oscillates with a lower frequency than $e^{i(\delta_1+\delta_2)/2}$. Since Z is a complex number it can be written in the form

$$Z = re^{i\theta}.$$

Determine an expression for r; this determines the amplitude of the envelope function.

c) Determine an expression for the time averaged irradiance provided that $1/\omega_j \ll T \ll 1/\Delta\omega$ where $\Delta\omega = \omega_1 - \omega_2$. Use this to determine the maximum and minimum irradiance as time passes.

2 Superposition of three waves

Consider three waves, all produced at the same source with the same phase and polarization.

$$\begin{aligned}\tilde{\mathbf{E}}_1 &= E_0 e^{i(k_1 x - \omega_1 t)}\hat{\mathbf{y}} \\ \tilde{\mathbf{E}}_2 &= E_0 e^{i(k_2 x - \omega_2 t)}\hat{\mathbf{y}} \\ \tilde{\mathbf{E}}_3 &= E_0 e^{i(k_3 x - \omega_3 t)}\hat{\mathbf{y}}\end{aligned}$$

where $k_2 = k_1 + \Delta k$ and $k_3 = k_1 + 2\Delta k$.

a) Letting $\phi_j := k_j x - \omega_j t$, show that $\phi_2 = \phi_1 + \Delta\phi$ and $\phi_3 = \phi_1 + 2\Delta\phi$ for an appropriate $\Delta\phi$. Use these to rewrite

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_1 + \tilde{\mathbf{E}}_2 + \tilde{\mathbf{E}}_3$$

in terms of ϕ_1 and $\Delta\phi$.

b) The previous result should give a standard type of series, for whose sum there is a simple algebraic formula. Use this to determine an expression for $\tilde{\mathbf{E}}$. This expression will contain terms of the form $e^{i\alpha} - 1$. Such a term can be written as $e^{i\alpha} - 1 = e^{i\alpha/2}(e^{i\alpha/2} - e^{-i\alpha/2}) = 2i\, e^{i\alpha/2} \sin(\alpha/2)$. Rewrite the expression for $\tilde{\mathbf{E}}$ using this result.

c) Show that the resulting wave is a product of a wave with wavenumber k_2 and an envelope of the form

$$\frac{\sin(3\Delta\phi/2)}{\sin(\Delta\phi/2)}.$$

d) Plot the envelope as a function of $\Delta\phi$ for $-5 \leqslant \Delta\phi \leqslant 5$ and use this to provide a qualitative sketch of the electric field.

e) Determine values of $\Delta\phi$ for which the envelope reaches a maximum and use this to determine the group velocity of these waves. *With some trigonometry and algebra you should be able to show that the condition for a maximum is* $\sin(\Delta\phi) = 0$.

3 Bennett, *Principles of Physical Optics*, 5.14, page 223.

4 Bennett, *Principles of Physical Optics*, 5.15, page 224.

5 Time evolution of a wave

Suppose that, at $t = 0$,

$$\psi(x, 0) = Ae^{-x^2/2\sigma^2}$$

where A and σ are real and positive.

a) Determine the Fourier transform

$$\tilde{\psi}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(x, 0) e^{-ikx} \mathrm{d}x.$$

b) Use the previous result to write a general expression, in integral form, for $\psi(x, t)$, at any time t.

c) Assume that $\omega = vk$ where v is independent of k. Substitute into the result from the previous part to determine $\psi(x, t)$. Describe the shape of the resulting wave and how this evolves as time passes.