Modern Optics: Homework 16

Due: 16 October 2015

1 Irradiance and complex field representations

The time-dependent irradiance is

$$I(t) = \varepsilon v \mathbf{E} \cdot \mathbf{E}$$

where $\mathbf{E}$ is the real electric field associated with a wave. The time-averaged irradiance at time t_0 is

$$\langle I \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} I(t) \mathrm{d}t$$

where T is an appropriately chosen duration for averaging. Mathematically it is more convenient to represent the wave using a complex electric field $\tilde{\mathbf{E}}$ and

$$\mathbf{E} = \mathrm{Re}\,[\tilde{\mathbf{E}}] = \frac{\tilde{\mathbf{E}} + \tilde{\mathbf{E}}^*}{2}. \tag{1}$$

a) Consider a single plane wave represented by

$$\tilde{\mathbf{E}} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

where $\mathbf{E}_0$ is real. Show that if $T \gg 1/\omega$ then

$$\langle I \rangle = \frac{1}{2}\,\varepsilon v \tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}^*.$$

b) Now consider a superposition of

$$\begin{aligned} \tilde{\mathbf{E}}_1 &= \mathbf{E}_{01} e^{i(\mathbf{k}_1\cdot\mathbf{r}_1 - \omega_1 t)} \\ \tilde{\mathbf{E}}_2 &= \mathbf{E}_{02} e^{i(\mathbf{k}_2\cdot\mathbf{r}_2 - \omega_2 t)}. \end{aligned}$$

Here

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_1 + \tilde{\mathbf{E}}_2.$$

Show that if $T \gg 1/\omega_1, T \gg 1/\omega_2$ then

$$\langle I \rangle = \frac{1}{2}\,\varepsilon v \left\langle \tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}^* \right\rangle.$$

c) Suppose that $\mathbf{E}_{01} = E_{01}\hat{\mathbf{y}}$ and $\mathbf{E}_{02} = E_{02}\hat{\mathbf{y}}$. Suppose that $\mathbf{k}_i = k_i\hat{\mathbf{x}}$, that sources 1 is located on the x-axis at x_1, source 2 is located on the x-axis at x_2, and the detector is is located on the x-axis at x. Show that if $T \ll 1/\Delta\omega$ where $\Delta\omega = \omega_1 - \omega_2$ then

$$\langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2\sqrt{\langle I_1 \rangle \langle I_2 \rangle}\, \cos\left(\Delta k x - \Delta\omega t + \phi\right)$$

where $\Delta k = k_1 - k_2$ and ϕ is independent of x and t. *Hint: This requires an approximation involving a Taylor series for an exponential.*

2 Bennett, *Principles of Physical Optics*, 5.9, page 218.

3 Bennett, *Principles of Physical Optics*, 5.10, page 218.