Modern Optics: Class Exam I

21 September 2015

Name: Solution **Total:** /50

Instructions

- There are 6 questions on 7 pages.
- Show your reasoning and calculations and always justify your answers.

Physical constants and useful formulae

$$c = 3.0 \times 10^{8}\,\mathrm{m/s} \qquad \epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2} \qquad \mu_0 = 4\pi \times 10^{-12}\,\mathrm{Tm/A}$$

Do 1 or 2

Question 1

In a vacuum, the electric field satisfies

$$\nabla^2 \mathbf{E} = \epsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \mathbf{E}.$$

a) Show by direct substitution, that

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)},$$

where $\mathbf{E}_0$ is constant, satisfies this equation provided that k and ω satsify a particular relationship. Provide the relationship.

$$\nabla^2 \vec{E} = \frac{\partial^2}{\partial x^2}\vec{E} + \frac{\partial^2}{\partial y^2}\vec{E} + \frac{\partial^2}{\partial z^2}\vec{E}$$

Now $\vec{E} = \vec{E}_0\, e^{i(k_x x + k_y y + k_z z - \omega t)}$

$$\Rightarrow \quad \frac{\partial^2}{\partial x^2}\vec{E} = -k_x^2 \vec{E}$$

Question 1 continued ...

Similarly

$$\frac{\partial^2 \vec{E}}{\partial y^2} = -k_y^2 \vec{E} \qquad \frac{\partial^2 \vec{E}}{\partial z^2} = -k^2 \vec{E}$$

$$\Rightarrow \nabla^2 \vec{E} = -\underbrace{(k_x^2 + k_y^2 + k_z)}_{k^2}{}^2 \vec{E} = -k^2 \vec{E}$$

Then $\dfrac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}$, Substituting

$$-k^2 \cancel{\vec{E}} = -\mu_0 \epsilon_0 \omega^2 \cancel{\vec{E}}$$

$$k^2 = \mu_0 \epsilon_0 \omega^2 \quad \text{or} \quad k^2 = \omega^2 / v^2$$

5

b) Use Maxwell's equations to show that the associated magnetic field is $\mathbf{B} = \frac{1}{\omega} \mathbf{k} \times \mathbf{E}$.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

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Now $\vec{\nabla} \times \vec{E} = \left(\dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} \right) \hat{x} + \dots$

$$= \left[+ik_y E_{z0} - ik_z E_{y0} \right] e^{i(\dots)} + \dots$$

$$= +i \vec{k} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \frac{\partial \vec{B}}{\partial t} = -i \vec{k} \times \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\Rightarrow \vec{B} = \frac{\vec{k} \times \vec{E}_0}{\omega} e^{i(\dots)} \Rightarrow \vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}$$

/~~8~~ 10

Question 2

A particular beam of light can be describe by a plane electromagnetic wave

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)},$$

where $\mathbf{E}_0$ is constant. The beam has a circular cross section with diameter 0.040 m and carries power 0.020 W.

a) Determine the magnitude of $\mathbf{E}_0$.

$$I = \frac{1}{2} c \epsilon_0 E_0^2 \quad \text{and} \quad I = P/A = \frac{P}{\pi r^2}$$

$$\Rightarrow \frac{P}{\pi r^2} = \frac{1}{2} c \epsilon_0 E_0^2 \Rightarrow E_0 = \sqrt{\frac{2P}{\pi r^2 c \epsilon_0}}$$

$$\Rightarrow E_0 = \sqrt{\frac{2 \times 0.020\,W}{\pi \times (0.020 m)^2 \times 3 \times 10^8 m/s \times 8.85 \times 10^{-12}\, C^2/Nm^2}}$$

$$= 109 N/C.$$

6

b) Suppose that the beam propagates along the positive x axis and that $\mathbf{E}_0 = E_0\hat{\mathbf{j}}$. Determine the direction of the magnetic field.

4

$$\vec{B} = \frac{1}{\omega} \vec{k} \times \vec{E}$$

$$\vec{k} = \hat{\imath} \Rightarrow \vec{B} = \frac{1}{\omega}(\hat{\imath} \times E_0 \hat{\jmath})$$

$$= \frac{E_0}{\omega} \hat{k}$$

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along z axis

Question 3

Light incident in silica (index of refraction 1.46) strikes a surface where silica meets air (index of refraction 1.00).

a) Determine the angle(s) of incidence (from the normal) at which no light passes into the air. beyond critical angle for total internal reflection, via

1.46

1.00

$n_t \sin 90° = n_i \sin\theta_c$.

$$\theta_c = \sin^{-1}\left(\frac{n_t}{n_i}\right) = \sin^{-1}\left(\frac{1}{1.46}\right) = 43.2° \Rightarrow \theta_i > 43.2°$$

b) Determine the angle of incidence (from the normal) at which it is possible that all the light is transmitted into the air and none reflected back into the silica.

Brewster's angle

$$\theta_t = \tan^{-1}\left(\frac{n_t}{n_i}\right) = \tan^{-1}\left(\frac{1}{1.46}\right) = \qquad \theta_i = 34.4°$$

/6 8

Question 4

$\sin\theta_t = \frac{n_i}{n_t}\sin\theta_i$

$\theta_t = 39°$

Light is incident in air (index of refraction 1.00) on a glass surface (index of refraction 1.50) at an angle of 70° from the normal. Describe how you would orient a polarizing filter (i.e. which way should its transmission axis point?) so as to eliminate most of the reflected light from the surface. Briefly explain your answer.

It is generally true that $R^{\perp} > R^{\parallel}$. Here $R^{\perp} \left(\frac{n_i\cos\theta_i - n_t\cos\theta_t}{\ldots + \ldots}\right)^2$

$$= \left(\frac{1.15}{1.84}\right)^2 = 0.39$$

and $R^{\parallel} = \left(\frac{n_i\cos\theta_t - n_t\cos\theta_i}{\ldots + \ldots}\right)^2 = \left(\frac{0.26}{1.29}\right)^2 = 0.04$

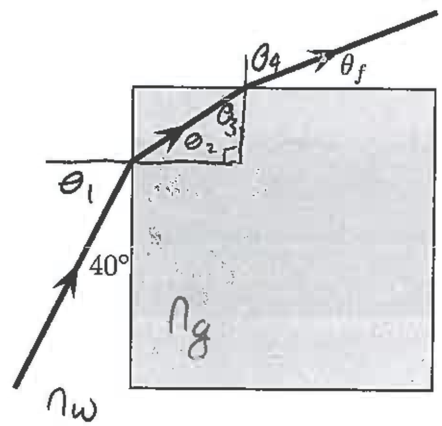
want to eliminate perpendicular pol

/3 4

$\Rightarrow$ orient transmission axis along plane of incidence

Question 5

Light in water (index of refraction 1.33) is incident on a square block of glass (index of refraction 1.50) as illustrated. It passes through the block, emerging in the water above the block.



a) Determine the illustrated angle, θ_f at which the beam emerges from the top surface of the block.

note $\theta_3 + \theta_2 = 90° \Rightarrow \theta_3 = 90° - \theta_2$

$n_w \sin\theta_1 = n_g \sin\theta_2$

$\Rightarrow \theta_2 = \sin^{-1}\left(\frac{n_w}{n_g} \sin\underbrace{\theta_1}_{50°}\right) = \sin^{-1}\left[\frac{1.33}{1.50} \sin 50°\right] = 43°$

$\Rightarrow \theta_3 = 47°$

$n_w \sin\theta_4 = n_g \sin\theta_3 \Rightarrow \sin\theta_4 = \frac{1.5}{1.33} \sin 47° = 0.828$

7

$\Rightarrow \theta_4 = 55.9° \Rightarrow \theta_f = 90 - 55.9° = 34.1°$

b) Suppose that the incident beam is polarized so that the electric field points out of the page. Determine the reflectivity and transmittivity at the left hand surface.

This is perp pol.

$$R^{\perp} = \left[\frac{n_i \cos\overset{\theta_1}{\theta_i} - n_t \cos\overset{\theta_2}{\theta_t}}{n_i \cos\theta_i + n_t \cos\theta_t}\right]^2$$

$$= \left[\frac{1.33 \cos 50° - 1.50 \cos 43°}{\ldots + \ldots}\right]$$

Question 5 continued . . .

$$R^{\perp} = \left[\frac{0.855 - 1.10}{0.855 + 1.10} \right]^2 = 0.015$$

$$T^{\perp} + R^{\perp} = 1 \Rightarrow T^{\perp} = 1 - R^{\perp}$$

$$= 0.985$$

6

c) Suppose that one required that no light emerge from the top surface, but that light does pass through the left side of the glass; this requires that the incident beam strike the left surface at an angle different to 40°. Should the angle between the incident beam and the left surface be larger than or smaller than 40°? Briefly explain your answer.

Need $\theta_3 > \theta_c = \sin^{-1}\left(\frac{n_w}{n_g}\right) = 63°$ 3

Means θ_2 must be reduced

$\Rightarrow \theta_1$ " " "

$\Rightarrow \theta_i$ must " increased

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Question 6

Starting with the Fresnel equations show that for light polarized parallel to the plane of incidence,

$$R^{\parallel} + T^{\parallel} = 1,$$

for any angle of incidence.

$$r^{\parallel} = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{\cdots + \cdots}$$

$$t^{\parallel} = \frac{2 n_i \cos\theta_i}{\cdots + \cdots}$$

Then $R^{\parallel} = (r^{\parallel})^2$ $\qquad T^{\parallel} = \frac{\cos\theta_t}{\cos\theta_i} \cdot \frac{n_t}{n_t} (t^{\parallel})^2$

$$R^{\parallel} + T^{\parallel} = \frac{(n_t \cos\theta_i - n_i \cos\theta_t)^2 + \frac{\cos\theta_t}{\cos\theta_i} \cdot \frac{n_t}{\cancel{n_i}} \cdot 4 n_i^{\cancel{2}} \cos^2\theta_i}{(n_t \cos\theta_i + n_i \cos\theta_t)^2}$$

$$= \frac{n_t^2 \cos^2\theta_i + n_i^2 \cos^2\theta_t - \cancel{2 n_i n_t \cos\theta_i \cos\theta_t} + \cancel{4}^{\,2} n_i n_t \cos\theta_i \cos\theta_t}{[\cdots + \quad]^2}$$

$$= \frac{(n_t \cos\theta_i + n_i \cos\theta_t)^2}{(n_t \cos\theta_i + n_i \cos\theta_t)^2} = 1$$

/~~10~~ 12