Phys 230 2013

Lecture 42

Fri: HW due

Read: 7.2

In general, sources will be positioned relative to each other so that neither perfect constructive nor destructive interference occurs. The more general case can be analyzed by obtaining expressions for the displacement produced by each wave which refer to the source location. Figure III.1.4 illustrates a source located at x_0.

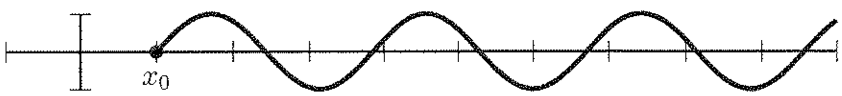


Figure III.1.4: General source.

Suppose that the source oscillates with frequency ω and amplitude A and that the wavespeed is v. The vertical displacement of the medium at the source location is

$$y_{\text{source}}(t) = A\cos(\omega t - \phi)$$

This disturbance takes time $(x - x_0)/v$ to reach any location, at position x, to the right of the source. Thus the displacement at x at time t is equal to that at the source at time $t - (x - x_0)/v$, or

$$\begin{aligned} y(x,t) &= A\cos\left[\omega(t - (x - x_0)/v) - \phi\right] \\ &= A\cos\left[\omega t - k(x - x_0) - \phi\right] \end{aligned}$$

where we have used $\omega = kv$. Rearranging gives:

> If a source at x_0 oscillates according to
>
> $$y_{\text{source}}(t) = A\cos(\omega t - \phi)$$
>
> then the displacement of the medium at any location $x > x_0$ at time t is
>
> $$y(x,t) = A\cos\left[k(x - x_0) - \omega t + \phi\right]$$
>
> where $k = \omega/v$ is the wavenumber of the waves with v being the wavespeed.

(III.1.3)

We can apply this to the overall disturbance produced by two sources. Suppose that these have the same amplitude, frequency and are in phase, i.e. k, ω and ϕ are the same for both. Thus, if source j is located at x_j, then the two individual disturbances are described by

$$\begin{aligned} y_1(x,t) &= A\cos\left[k(x - x_1) - \omega t + \phi\right] \\ y_2(x,t) &= A\cos\left[k(x - x_2) - \omega t + \phi\right]. \end{aligned}$$

This implies that the disturbance on the medium is

$$y(x,t) = A\cos\left[k(x - x_1) - \omega t + \phi\right] + A\cos\left[k(x - x_2) - \omega t + \phi\right] \qquad \text{(III.1.4)}$$

This can be simplified by using a complex representation for the two waves. This requires a complex function $z(x, t)$ such that

$$y(x,t) = \mathrm{Re}[z(x,t)]. \tag{III.1.5}$$

Here the appropriate choices for the individual disturbances are of the form

$$z_j(x,t) = Ae^{i(k(x-x_j)-\omega t+\phi)}. \tag{III.1.6}$$

Thus the combined disturbance is represented by

$$\begin{aligned}
z(x,t) &= Ae^{i(k(x-x_1)-\omega t+\phi)} + Ae^{i(k(x-x_2)-\omega t+\phi)} \\
&= Ae^{i(kx-\omega t+\phi)}\left[e^{-ikx_1} + e^{-ikx_2}\right] \qquad \text{quiz 1.} \\
&= Ae^{i(kx-\omega t+\phi)}e^{-ik(x_1+x_2)/2}\left[e^{-ik(x_1-x_2)/2} + e^{ik(x_1-x_2)/2}\right] \\
&= Ae^{i(kx-\omega t+\phi)}e^{-ik(x_1+x_2)/2}2\cos\left[\frac{k(x_1-x_2)}{2}\right] \\
&= 2A\cos\left[\frac{k(x_1-x_2)}{2}\right]e^{i[k(x-x_1/2-x_2/2)-\omega t+\phi]}
\end{aligned} \tag{III.1.7}$$

Finally Eq. (III.1.5) gives

$$y(x,t) = 2A\cos\left(\frac{k\Delta x}{2}\right)\cos\left[k\left(x - \frac{x_1}{2} - \frac{x_2}{2}\right) - \omega t + \phi\right]. \tag{III.1.8}$$

where $\Delta x = x_1 - x_2$. The factor

$$2A\cos\left(\frac{k\Delta x}{2}\right)$$

only refers to the location of the two sources and not any point along the the medium. The remaining factor

$$\cos\left[k\left(x - \frac{x_1}{2} - \frac{x_2}{2}\right) - \omega t + \phi\right] = \cos\left[kx - \omega t + \phi'\right],$$

where $\phi' = \phi - \frac{kx_1}{2} - \frac{kx_2}{2}$, represents a sinusoidal wave traveling with speed $v = \omega/k$ to the right. Thus these wave interfere to produce a superposition with amplitude $2A\cos\left(\frac{k\Delta x}{2}\right)$.

Summarizing:

> If two sources each oscillate with the same amplitude, A, the same frequency, ω and are in phase, then they interfere to produce a traveling sinusoidal wave with the same frequency as the individual sources and with amplitude
>
> $$2A\left|\cos\left(\frac{k\Delta x}{2}\right)\right| \tag{III.1.9}$$
>
> where Δx is distance between the two sources and $k = \omega/v$ is the wavenumber of the sources.

Quiz 2

Note that constructive interference occurs when

$$\cos\left(\frac{k\Delta x}{2}\right) = \pm 1$$

or

$$\frac{k\Delta x}{2} = n\pi \Rightarrow \Delta x = \frac{2n\pi}{k}$$

$$\Rightarrow \Delta x = \frac{2n\pi}{2\pi/\lambda} = n\lambda$$

as predicted before. Then destructive interference arises when

$$\cos\left(\frac{k\Delta x}{2}\right) = 0 \Rightarrow \frac{k\Delta x}{2} = \frac{2n+1}{2}\pi$$

$$\Rightarrow k\Delta x = (2n+1)\pi$$

$$\Rightarrow \Delta x = \frac{(2n+1)\pi}{k} = \frac{(2n+1)\cancel{\pi}}{2\cancel{\pi}/\lambda} = \frac{2n+1}{2}\lambda$$

$$\Rightarrow \Delta x = (n+\tfrac{1}{2})\lambda.$$

We can even determine when the amplitude of the superposition is A. Here we require

$$\cos\left(\frac{k\Delta x}{2}\right) = \pm\frac{1}{2} \Rightarrow \frac{k\Delta x}{2} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \text{ etc...}$$

$$\Rightarrow \frac{\cancel{2}\pi}{\lambda}\frac{\Delta x}{\cancel{2}} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

$$\Rightarrow \Delta x = \frac{\lambda}{3}, \frac{2\lambda}{3}, \frac{4\lambda}{3}, \ldots$$

These are the results that we obtained previously by more qualitative considerations.

Double slit interference.

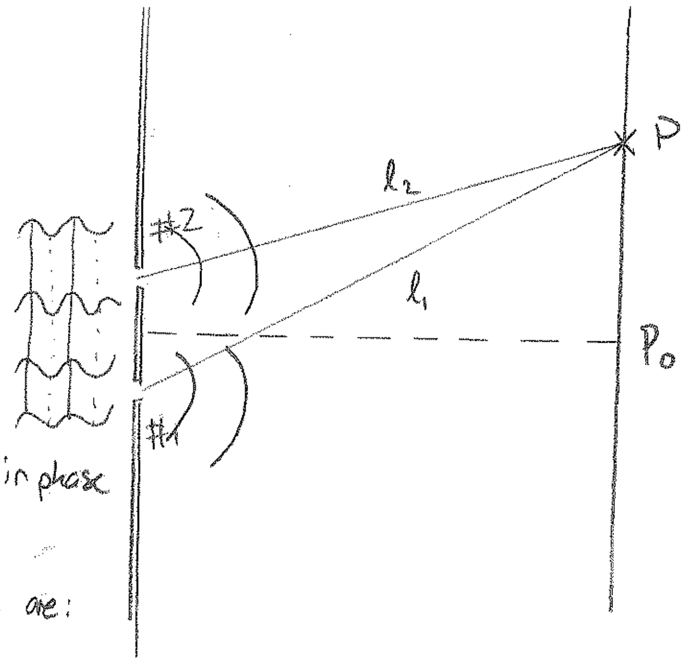
A typical example of interference arises when a wave is incident on a barrier with two openings

Initially we assume that

1) the width of the slits is negligible,

2) each slit produces a wave that radiates isotropically in a circular fashion

3) the disturbances at the slits are in phase



Then these two waves overlap and interfere. What we would like are:

1) the amplitude of the combined wave at any point P on the screen.

2) the power arriving at any point P on the screen.

Then the disturbance at P from slit 1 is:

$$A\cos(kl_1 - \omega t + \phi)$$

and from slit 2

$$A\cos(kl_2 - \omega t + \phi)$$

By the rule that we obtained for the superposition from both sources

1) the disturbance at P oscillates with frequency ω.

2) the disturbance at P has amplitude

$$2A\left|\cos\left(\frac{k\Delta l}{2}\right)\right|$$

where $\Delta l = l_1 - l_2$.

The power transmitted by any wave is

$$P = 2\alpha v \omega^2 B^2$$

where B is the amplitude of the wave and v the wavespeed. Here κ is a constant that depends on the physical situation. Thus we find that

$$P = 2\alpha v \omega^2 \, 4A^2 \cos^2\left(\frac{k\Delta l}{2}\right)$$

Intensity is proportional to power and so

$$\boxed{I = \beta \cos^2\left(\frac{k\Delta l}{2}\right)}$$

where β is a constant.

Quiz 3

Denote the intensity at P_0 by I_0. For this, $\Delta l = 0$. Thus $I_0 = \beta$. So we get

$$\boxed{I = I_0 \cos^2\left(\frac{k\Delta l}{2}\right)}$$

We get local maxima when $\cos^2\left(\frac{k\Delta l}{2}\right) = 1$. Thus we get local maxima when.

$$\frac{k\Delta l}{2} = n\pi \qquad n = 0, 1, 2, 3 \ldots$$

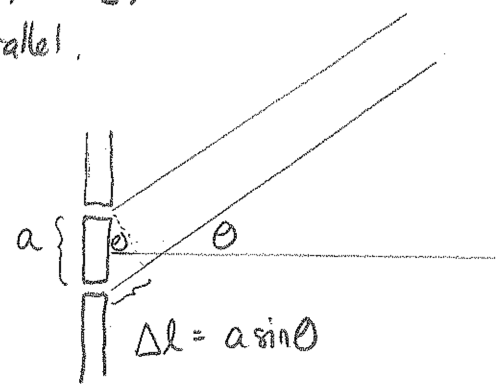
$$\Rightarrow \frac{2\cancel{\pi}}{\lambda}\frac{\Delta l}{\cancel{2}} = n\cancel{\pi} \quad \Rightarrow \quad \Delta l = n\lambda$$

So

$$\boxed{\begin{array}{l}\text{local maxima of intensity occur when} \\ \Delta l = n\lambda \qquad n = 0, \pm 1, \pm 2, \ldots\end{array}}$$

Distant screen approximation.

If the screen is distant compared to the slit spacing, a, then the two waves leave the slits approximately parallel.



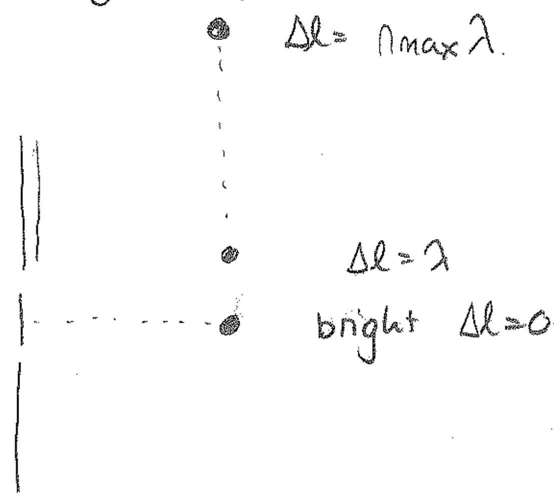
Then

$$\Delta l \cong a \sin\theta$$

So

$$I = I_0 \cos^2\left[\frac{ka}{2}\sin\theta\right]$$

Exercise: Determine the maximum number of bright fringes that could occur for a given slit spacing.



Answer: Bright fringes occur when

$$\Delta l = \pm n\lambda$$

But in the $\theta = 90^\circ$ case

$$\Delta l = a$$

So we get $\quad \Delta l \leq a$

$$\Rightarrow \quad n\lambda \leq a \quad \Rightarrow \quad n \leq a/\lambda.$$

Thus n_{max} is the largest integer less than a/λ. Then the number of bright fringes is $2n_{max}+1$.