Mon Read

HW

Waves on a medium of fixed length

In general waves on a medium of fixed length are described via

Wave equation
$$\frac{\partial^2 y}{\partial t^2} = V^2 \frac{\partial^2 y}{\partial x^2}$$
+ Boundary conditions at ends

For example, on a string of length L which extends from $x=0$ to $x=L$, the boundary conditions are: $y(0,t)=0$ and $y(L,t)=0$. The differential equation can then be solved in such a way that the solutions respect these boundary conditions.

We shall illustrate this by considering a string with both ends fixed. Thus we require

$$y(0,t) = 0$$
$$y(L,t) = 0$$

The technique of separation of variables then yielded certain special solutions called standing waves

The process is:

Wave equation:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial t^2}$$

+

Boundary conditions

$$y(0,t) = 0$$
$$y(L,t) = 0$$

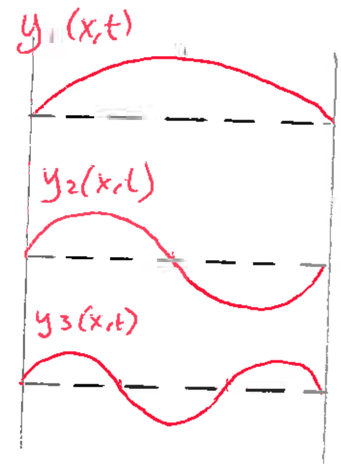
$\Downarrow$ separation of variables (x,t)

Standing wave solutions

$$y_n(x,t) = A_n \sin(k_n x)\cos(\omega_n t + \phi_n)$$

where $k_n = \frac{n\pi}{L}$ and A_n, ϕ_n are constants

$\omega_n = k_n v$



Each standing wave has the property that all points on the wave oscillate with the same frequency.

In these standing waves, the constants A_n and ϕ_n are determined by initial conditions at all locations along the string. Specifically

$$\boxed{\begin{aligned} y_n(x,0) &= A_n \sin(k_n x)\cos(\phi_n) \\ \frac{\partial y_n}{\partial t}(x,0) &= -\omega_n A_n \sin(k_n x)\sin(\phi_n) \end{aligned}}$$

Quiz 1

We shall consider situations where at $t=0$, the string is at rest.

So

> If the string is at rest at $t=0$ then the possible standing wave solutions have form:
>
> $$y_n(x,t) = A_n \sin(k_n x)\cos(\omega_n t)$$
>
> where $k_n = \frac{n\pi}{L}$ and $\omega_n = k_n v$ with $n = 1, 2, 3, \ldots$

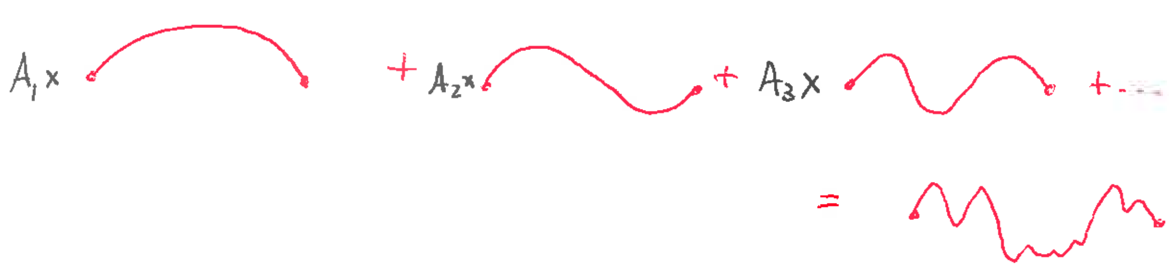
Superpositions of standing waves.

A general mathematical theorem states that:

> If $y_1(x,t)$ and $y_2(x,t)$ are any solutions to the wave equation then
>
> $$y(x,t) = \alpha_1 y_1(x,t) + \alpha_2 y_2(x,t)$$
>
> is also a solution to the wave equation.

We can iterate this to attain a solution to the wave equation that is more general than that offered by a single standing waves. Quite generally we can make an infinite superposition. If the string is initially at rest a general solution is represented by

> $$y(x,t) = \sum_{n=1} A_n \sin(k_n x)\cos(\omega_n t)$$
>
> where $k_n = \frac{n\pi}{L}$, $\omega_n = \frac{n\pi}{L} v$



Fourier analysis

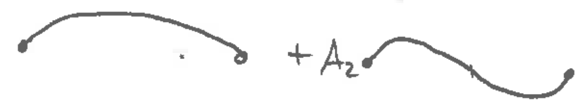
We have seen how we can construct general superpositions on a string with fixed ends by adding standing waves. We now address the reverse question:

Given a disturbance on the string at $t=0$

i.e. given $y(x,0)$

=

Can we express this as a superposition of standing waves?

A_1  $+ A_2$ ...

$+ \cdots$

i.e. can we find A_n so that

$$y(x,0) = \sum A_n \sin(k_n x)\cos(\omega_n 0)$$
$$= \sum A_n \sin(k_n x)$$

<u>Fourier analysis</u> addresses this issue. The series:

$$\boxed{y(x,0) = \sum_{n=1}^{\infty} A_n \sin(k_n x)} \qquad \left(k_n = \frac{n\pi}{L}\right)$$

is called a <u>Fourier series</u>. The coefficients A_n are called <u>Fourier coefficients</u>. We can show that

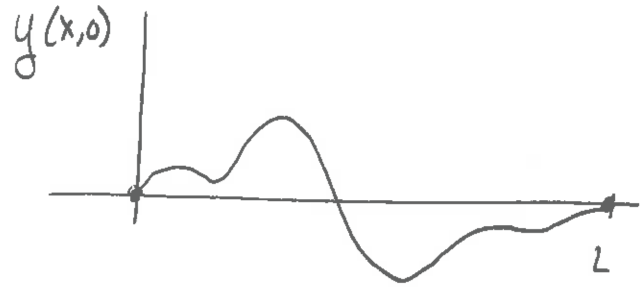
Given a known initial displacement, $y(x,0)$, the Fourier coefficients in $y(x,0) = \sum A_n \sin\left(\frac{n\pi x}{L}\right)$ are determined via:

$$A_n = \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$$

for $n = 1, 2, 3, \ldots$

Note that this also determines the temporal behavior of the string. The scheme is:

A string has fixed ends at $x=0$ and $x=L$. The string is initially at rest and has displacement $y(x,0)$



Fourier analysis suggests:

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin(k_n x) \cos(\omega_n t)$$

any t   any t

$k_n = \frac{n\pi}{L}$

$\omega_n = \frac{n\pi}{L} v$

with

$$A_n = \frac{2}{L} \int_0^L y(x,0) \sin(k_n x)\, dx$$

Maple sheet:

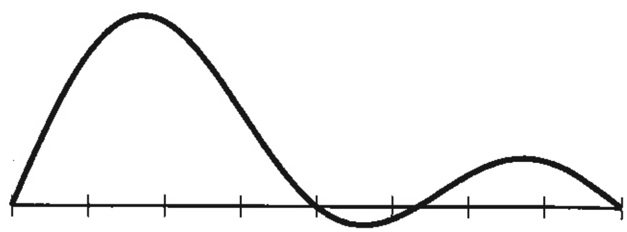
teaching/topics/mechanics/waves/maplework/fixedstring.mw

Intermediate Dynamics: Group Exercises 11

Superpositions of Standing Waves

1 Waves on a string with fixed ends.

A string has fixed ends at $x = 0$ and $x = L$. In one particular situation, a snapshot of the string at $t = 0\,\mathrm{s}$ is illustrated.



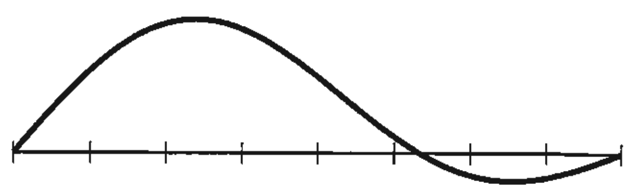
a) Does this satisfy the boundary conditions for a string with fixed ends? Does it resemble any of the known standing waves for a string with both ends fixed?

b) One possibility for describing such situations is via linear combinations, or superpositions, of standing waves. Consider

$$y(x,t) = A_1 \sin(k_1 x)\cos(\omega_1 t) + A_2 \sin(k_2 x)\cos(\omega_2 t)$$

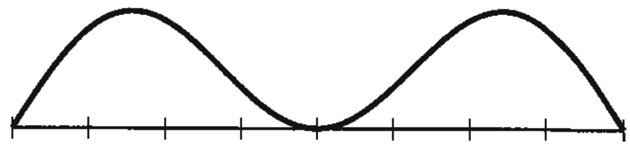
where $k_n = n\pi/L$ and $\omega_n = k_n v$. Show that this satisfies the wave equation and the boundary conditions for a string with both ends fixed, regardless of the values of A_1 and A_2. Consider the displacement of the string at $x = L/4$. Does this oscillate with one frequency or not?

c) Consider the string whose initial displacement is as illustrated.



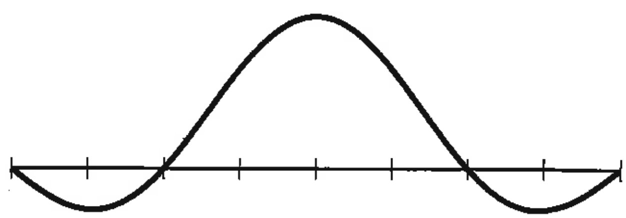
This could possibly be represented by superposition of standing waves. Suggest which standing waves for a string with both ends fixed might be able to produce the illustrated pattern.

d) Consider the string whose initial displacement is as illustrated.



This could possibly be represented by superposition of standing waves. Suggest which standing waves for a string with both ends fixed might be able to produce the illustrated pattern.

e) Consider the string whose initial displacement is as illustrated.



This could possibly be represented by superposition of standing waves. Suggest which standing waves for a string with both ends fixed might be able to produce the illustrated pattern.

Answer:

a) It satisfies the boundary conditions since clearly $y(0,t) = 0$ and $y(L,t) = 0$. It does not resemble a sinusoidal function.

b) First

$$\frac{\partial^2 y}{\partial t^2} = -\omega_1^2 A_1 \sin(k_1 x)\cos(\omega_1 t) - \omega_2^2 A_2 \sin(k_2 x)\cos(\omega_2 t).$$

Then

$$v^2 \frac{\partial^2 y}{\partial x^2} = -v^2 k_1^2 A_1 \sin(k_1 x)\cos(\omega_1 t) - v^2 k_2^2 A_2 \sin(k_2 x)\cos(\omega_2 t).$$

Now $\omega_i = k_i v$ gives

$$\frac{\partial^2 y}{\partial t^2} = -v^2 k_1^2 A_1 \sin(k_1 x)\cos(\omega_1 t) - v^2 k_2^2 A_2 \sin(k_2 x)\cos(\omega_2 t) = v^2 \frac{\partial^2 y}{\partial x^2}.$$

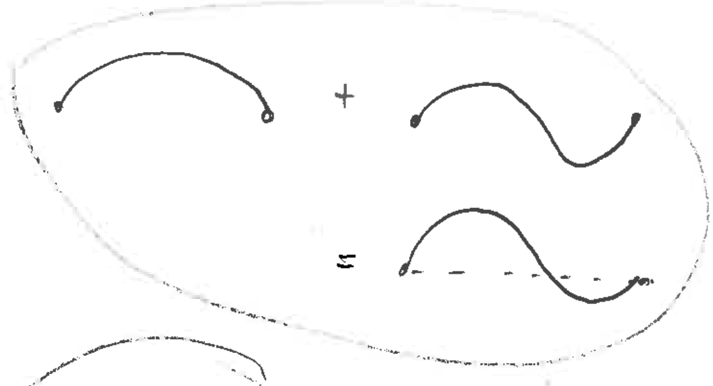
So $y(x,t)$ satisfies the wave equation.

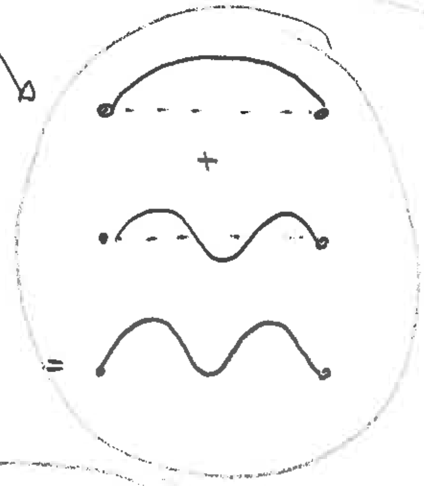
Simple substitution shows that $y(0,t) = y(L,t) = 0$.

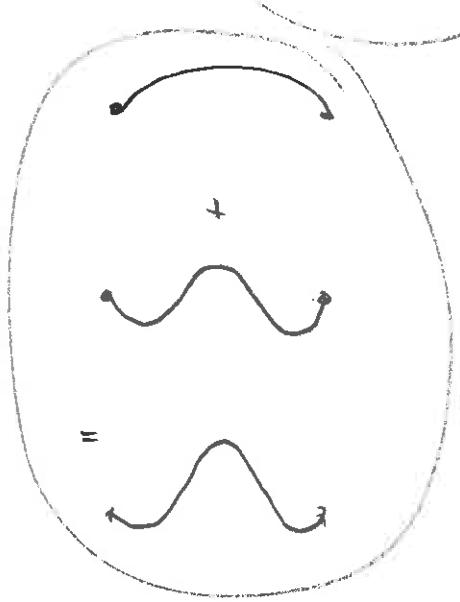
c) Probably $y_1(x,t)$ *and* $y_2(x,t)$

d) Probably $y_1(x,t)$ *and* $y_3(x,t)$

e) Probably $y_1(x,t)$ *and* $-y_3(x,t)$

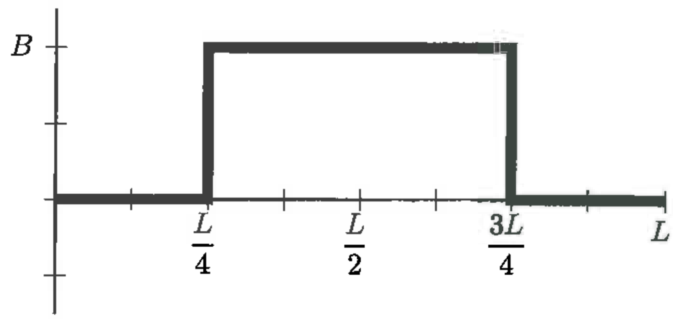






2 Fourier Analysis

A string has fixed ends at $x = 0$ and $x = L$. The displacement along the string at $t = 0\,\mathrm{s}$ is illustrated below



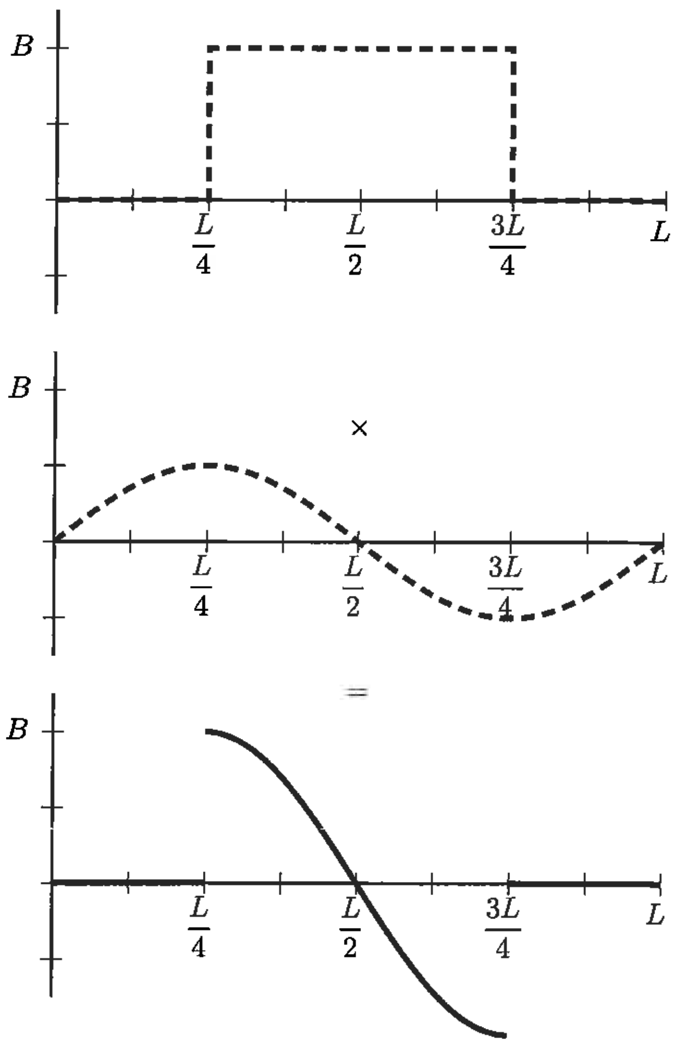
Determine the Fourier coefficients for this.

Answer:

The general rule for Fourier coefficients is

$$A_n = \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

Inspection suggest that when n is even the coefficient will be zero. For example, with $n = 2$,


B
L/4
L/2
3L/4
L
×
=

The area under the solid curve is clearly zero. In general

$$
\begin{aligned}
A_n &= \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{2}{L}\int_{L/4}^{3L/4} B\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= -\frac{2B}{L}\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{L/4}^{3L/4} \\
&= -\frac{2B}{n\pi}\left[\cos\left(\frac{n\pi 3L}{L4}\right) - \cos\left(\frac{n\pi 3L}{L4}\right)\right] \\
&= \frac{2B}{n\pi}\left[\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{n\pi 3}{4}\right)\right].
\end{aligned}
$$

When n is even the arguments of the cosine functions are odd multiples of $\pi/2$ and thus return zero. A table of the lowest few values is

n	A_n
1	$\frac{4B}{\sqrt{2}\pi}$
2	0
3	$-\frac{1}{3}\frac{4B}{\sqrt{2}\pi}$
4	0
5	$\frac{1}{5}\frac{4B}{\sqrt{2}\pi}$