Phys 230 2013

Lecture

Monday : HW due

Mon: 6.4

Energy in Standing Waves

The energy density for waves on a string is:

$$dE = \frac{1}{2}\left[\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right]$$

$$= \frac{1}{2}\mu\left[\left(\frac{\partial y}{\partial t}\right)^2 + v^2\left(\frac{\partial y}{\partial x}\right)^2\right]$$

Thus the energy along a string that extends from $x=0$ to $x=L$ is

$$E = \frac{1}{2}\mu\int_0^L\left[\left(\frac{\partial y}{\partial t}\right)^2 + v^2\left(\frac{\partial y}{\partial x}\right)^2\right]dx$$

The general solution for a standing wave is:

$$y(x,t) = \left[A\sin(kx) + B\cos(kx)\right]\cos(\omega t+\phi)$$

and we can show:

$$\boxed{E = \frac{1}{4}\mu L\omega^2\left[A^2+B^2\right] + \frac{1}{4}\mu\omega v\left[(A^2-B^2)\cos(kL) - 2AB\sin(kL)\right]\sin(kL)\cos(\omega t+\phi)}$$

Proof

$$E = \frac{1}{2}\mu \int_0^L \left[\left(\frac{\partial y}{\partial t}\right)^2 + v^2\left(\frac{\partial y}{\partial x}\right)^2\right] dx$$

$$\frac{\partial y}{\partial t} = -\omega\left[A\sin(kx) + B\cos(kx)\right]\sin(\omega t + \phi)$$

$$\Rightarrow \int_0^L \left(\frac{\partial y}{\partial t}\right)^2 = \omega^2 \sin^2(\omega t + \phi)\int_0^L \left\{A^2\sin^2(kx) + 2AB\sin(kx)\cos(kx) + B^2\cos^2(kx)\right\} dx$$

$$= \omega^2\sin^2(\omega t + \phi)\left\{\int_0^L \frac{1}{2}A^2(1 - \cos(2kx))\,dx + \int_0^L AB\sin(2kx)\,dx + \int_0^L \frac{1}{2}B^2(1 + \cos(2kx))\,dx\right\}$$

$$= \omega^2\sin^2(\omega t + \phi)\left\{\frac{A^2}{2}\left[L - \frac{1}{2k}\sin(2kL)\right] - \frac{AB}{2k}\left[\cos(2kL) - 1\right] + \frac{B^2}{2}\left[L + \frac{1}{2k}\sin(2kL)\right]\right\}$$

$$= \frac{1}{2}\omega^2\sin^2(\omega t + \phi)\left\{(A^2 + B^2)L + \frac{AB}{k} - \frac{AB}{k}\cos(2kL) + \frac{B^2 - A^2}{2k}\sin(2kL)\right\}$$

$$\frac{\partial y}{\partial x} = k\left[A\cos(kx) - B\sin(kx)\right]\cos(\omega t + \phi)$$

$$\Rightarrow v^2\left(\frac{\partial y}{\partial x}\right)^2 = \omega^2\left[A\cos(kx) - B\sin(kx)\right]^2\cos^2(\omega t + \phi)$$

By similar reasoning

$$\int_0^L v^2\left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{1}{2}\omega^2\cos^2(\omega t+\phi)\left\{(B^2+A^2)L - \frac{AB}{k} + \frac{AB}{k}\cos(2kL) - \frac{B^2-A^2}{2k}\sin(2kL)\right\}$$

So

$$E = \frac{1}{4}\mu\omega^2\left\{(A^2+B^2)L + \left[\frac{AB}{k} - \frac{AB}{k}\cos(2kL) + \frac{B^2-A^2}{2k}\sin(2kL)\right]\underbrace{\left(\sin^2(\omega t+\phi) - \cos^2(\omega t+\phi)\right)}_{=-\cos(2\omega t+2\phi)}\right\}$$

$$= \frac{1}{4}\mu\omega^2 L(A^2+B^2) + \frac{1}{4k}\mu\omega^2\left[AB(\underbrace{\cos(2kL)}_{1-2\sin^2(kL)}-1) + \frac{A^2-B^2}{2}\sin(2kL)\right]\cos(2\omega t+2\phi)$$

$$= \frac{1}{4}\mu\omega^2 L(A^2+B^2) + \frac{1}{4k}\mu\omega^2\left[AB(-2\sin^2(kL)) + (A^2-B^2)\sin(kL)\cos(kL)\right]\cos(2\omega t+2\phi)$$

$$= \frac{1}{4}\mu\omega^2 L(A^2+B^2) + \frac{1}{4}\mu\frac{\omega}{k}\omega\left[(A^2-B^2)\cos(kL) - 2AB\sin(kL)\right]\sin(kL) \times\cos(2\omega t+2\phi)$$

$$= \frac{1}{4}\mu\omega^2 L(A^2+B^2) + \frac{1}{4}\mu\omega v\left[(A^2-B^2)\cos(kL) - 2AB\sin(kL)\right]\sin(kL)\cos(\ldots)$$

We examine various boundary conditions:

$y(0,t)$	$y(L,t)$	$\left.\frac{\partial y}{\partial x}\right\|_0$	$\left.\frac{\partial y}{\partial x}\right\|_L$		E
0	0		—	$B=0 \quad k=\frac{n\pi}{L}$	$\frac{1}{4}\mu\omega^2 L A^2$
0	—	—	0	$B=0 \quad k=\frac{(2n+1)\pi}{L}$	$\frac{1}{4}\mu\omega^2 L A^2$
—	—	0	0	$A=0 \quad k=\frac{n\pi}{L}$	$\frac{1}{4}\mu\omega^2 L B^2$

In general, if the power flow at both ends is zero, then we expect energy is constant. This is true if

$$\left(\frac{\partial y}{\partial t}\frac{\partial y}{\partial x}\right)_{x=0} = 0 \qquad \text{and} \qquad \left(\frac{\partial y}{\partial t}\frac{\partial y}{\partial x}\right)_{x=L} = 0$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$-\omega\sin(\omega t+\phi)\,B\,kA\cos(\omega t+\phi) = 0 \qquad -k\omega\sin(\omega t+\phi)\cos(\omega t+\phi)\left[A\cos(kL) - B\sin(kL)\right] \times \left[A\sin(kL) + B\cos(kL)\right] = 0$$

$$\Rightarrow AB = 0 \qquad\qquad \Rightarrow (A^2-B^2)\sin(kL)\cos(kL) = 0$$

$$\Downarrow$$

$$E = \frac{1}{4}\mu L^2\omega^2\left[A^2+B^2\right] + \frac{1}{4}\mu\omega v\left[A^2-B^2\right]\cos(kL)\sin(kL)\cos(2\omega t+2\phi)$$

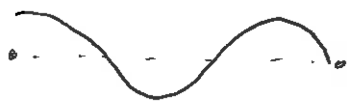
$$\Downarrow$$

$$E = \frac{1}{4}\mu L^2\omega^2(A^2+B^2)$$

Q1 a)

$$\frac{1}{4}\lambda = L \Rightarrow \lambda = 4L$$

$$\frac{3}{4}\lambda = L \Rightarrow \lambda = \frac{4L}{3}$$



$$\frac{5}{4}\lambda = L \Rightarrow \lambda = \frac{4L}{5}$$

general rule $\lambda = \frac{4L}{2n-1}$ $\quad n = 1, 2, 3, \ldots$

b) $$\frac{\partial y}{\partial x} = k\left[A\cos(kx) - B\sin(kx)\right]\cos(\omega t + \phi)$$

$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = 0 \Rightarrow k A\cos(\omega t + \phi) = 0 \Rightarrow A = 0$$

$$\Rightarrow y(x,t) = B\cos(kx)\cos(\omega t + \phi)$$

$$y(L,t) = 0 \Rightarrow \cos(kL) = 0 \Rightarrow kL = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$kL = \frac{\pi}{2}(2n-1) \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow k_n = \frac{\pi}{2L}(2n-1)$$

c) $$\frac{2\cancel{\pi}}{\lambda_n} = \frac{\cancel{\pi}}{2L}(2n-1) \Rightarrow \lambda_n = \frac{4L}{2n-1}$$

$$f_n \lambda_n = v \Rightarrow f_n = v/\lambda_n = \frac{v}{4L}(2n-1)$$

$$\omega_n = 2\pi f_n = \frac{v\pi}{2L}(2n-1)$$

2 Energy of a normal mode of a string with both ends fixed

Consider a string of length L with both ends fixed.

a) Determine an expression for the energy of the n^{th} normal mode.
b) Rewrite this in terms of the total mass of the string.

Answer:

a) The energy density for any wave is

$$\frac{1}{2}\left[\mu\left(\frac{\partial y}{\partial t}\right)^2+T\left(\frac{\partial y}{\partial x}\right)^2\right]=\frac{1}{2}\,\mu\left[\left(\frac{\partial y}{\partial t}\right)^2+v^2\left(\frac{\partial y}{\partial x}\right)^2\right].$$

Thus the energy along the string, with ends at $x=0$ *and* $x=L$ *is*

$$E=\frac{1}{2}\,\mu\int_0^L\left[\left(\frac{\partial y}{\partial t}\right)^2+v^2\left(\frac{\partial y}{\partial x}\right)^2\right]\,\mathrm{d}x.$$

The standing wave solution is

$$y_n(x,t)=A_n\sin(k_nx)\cos(\omega_nt+\phi).$$

It follows that for the mode labeled n,

$$\begin{aligned}
E_n&=\frac{1}{2}\,\mu\int_0^L\left[\left(\frac{\partial y_n}{\partial t}\right)^2+v^2\left(\frac{\partial y_n}{\partial x}\right)^2\right]\,\mathrm{d}x\\
&=\frac{1}{2}\,\mu A_n^2\int_0^L\left[\left(-\omega_n\sin(k_nx)\sin(\omega_nt+\phi)\right)^2+v^2\left(k_n\cos(k_nx)\cos(\omega_nt+\phi)\right)^2\right]\,\mathrm{d}x\\
&=\frac{1}{2}\,\mu A_n^2\int_0^L\left[\omega_n^2\sin^2(k_nx)\sin^2(\omega_nt+\phi)+v^2k_n^2\cos^2(k_nx)\cos^2(\omega_nt+\phi)\right]\,\mathrm{d}x\\
&=\frac{1}{2}\,\mu A_n^2\omega_n^2\int_0^L\left[\sin^2(k_nx)\sin^2(\omega_nt+\phi)+\cos^2(k_nx)\cos^2(\omega_nt+\phi)\right]\,\mathrm{d}x
\end{aligned}$$

since $\omega_n=vk_n$. *Then*

$$\begin{aligned}
\int_0^L\sin^2(k_nx)\mathrm{d}x&=\int_0^L\frac{1}{2}\left[1-\cos(2k_nx)\right]\mathrm{d}x\\
&=\frac{1}{2}\left[x-\frac{1}{2k_n}\sin(2k_nx)\right]\Bigg|_0^L\\
&=\frac{1}{2}\left[L-\frac{1}{2k_n}\sin(2k_nL)\right]\\
&=\frac{1}{2}\left[L-\frac{1}{2k_n}\sin(2n\pi L/L)\right]\\
&=\frac{1}{2}\left[L-\frac{1}{2k_n}\sin(2n\pi)\right]\\
&=\frac{L}{2}.
\end{aligned}$$

Similarly

$$\int_0^L \cos^2(k_n x)\mathrm{d}x = \frac{L}{2}.$$

Thus

$$\begin{aligned} E_n &= \frac{1}{4}\,\mu A_n^2 \omega_n^2 L\left[\sin^2(\omega_n t + \phi) + \cos^2(\omega_n t + \phi)\right] \\ &= \frac{1}{4}\,\mu A_n^2 \omega_n^2 L. \end{aligned}$$

b) The total mass of the string is $M = \mu L$ *and thus*

$$E = \frac{1}{4}\,A_n^2 M \omega_n^2$$

Note that the energy for an oscillator of mass m*, angular frequency* ω *and amplitude* A *is* $\frac{1}{2}\,A^2 m\omega^2$.