Phys 230 2013

Lecture 38

HW: Monday

SLINKY

Fri Ch 6.3, 6.4

So far we have considered waves that propagate along a medium without any ends. When ends are present waves will be reflected. This can be relatively simple to understand if the wave is a pulse. But if the wave approaches the boundary

Demo Slinky + pulse

continuously, then it will be reflected continuously and the incident + reflected waves will interfere. We need different techniques for describing such waves

Demo PhET W.O.S * Fixed end

* Tension 3/10

* Freq. 7 (non normal mode)

We observe what appears to be a complicated pattern. The strategy for understand this will be:

1) Find simple types of patterns (standing waves)

2) Show that any disturbance can be constructed in a simple way from standing waves

Demo: Slinky - standing waves

Demo Falstad animation - show normal modes

Standing waves on a string with fixed ends

To illustrate the typical analysis of waves in media of finite length, we consider a string with both ends fixed. Observation indicates that various simple wave forms can exist. An example, which can be produced readily on a slinky is illustrated.

Features of this case:

1) the pattern does not appear to move left or right - it flips up + down. This is called a standing wave.

2) Each point on the string oscillates up and down. The frequencies of oscillation are all the same.

3) The wavelength can be determined from a snapshot. The wavelength for the illustrated case happens to equal the length of the string. There are other possibilities though.

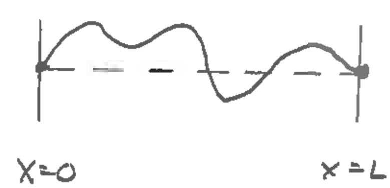
Mathematically these types of waves can be described by:

The string obeys the wave equation: $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$	+	The string is constrained at its ends $y(x=0,t) = 0$ $y(x=L,t) = 0$

This is called a boundary value problem.

Boundary value problems can be solved by a technique called separation of variables. This involves seeking solutions of the form:

$$y(x,t) = f(x)\,g(t)$$



We seek solutions for which $g(t) = \cos(\omega t + \phi)$ or

$$\boxed{y(x,t) = f(x)\cos(\omega t+\phi).}$$

These will have the same frequency at all points. The first task is to determine the form of $f(x)$. To do this, substitute into the wave equation. This gives:

$$\frac{\partial y}{\partial t} = f(x)(-\omega)\sin(\omega t+\phi) \qquad \frac{\partial^2 y}{\partial t^2} = -\omega^2 f(x)\cos(\omega t+\phi)$$

$$\frac{\partial y}{\partial x} = \frac{df}{dx}\cos(\omega t+\phi) \qquad \frac{\partial^2 y}{\partial x^2} = \frac{d^2 f}{dx^2}\cos(\omega t+\phi)$$

So

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2} \Rightarrow -\omega^2 f(x)\cos(\omega t+\phi) = v^2\frac{d^2 f}{dx^2}\cos(\omega t+\phi)$$

$$\Rightarrow \frac{d^2 f}{dx^2} = -\frac{\omega^2}{v^2} f = -k^2 f$$

where $k = \omega/v$. Thus

$$f(x) = A\sin(kx) + B\cos(kx)$$

and we have found a possible solution.

$$\boxed{y(x,t) = [A\sin(kx) + B\cos(kx)]\cos(\omega t+\phi) \quad \text{with } k=\omega/v} \quad -(1)$$

Quiz 1 ↳ true regardless of boundary conditions

Although (1) yields a solution to the wave equation, it has not used any information about the boundaries. The next step is to impose/apply the boundary conditions:

$$\boxed{\text{For all } t, \quad \begin{aligned} y(0,t) &= 0 \\ y(L,t) &= 0 \end{aligned}}$$

So $y(0,t) = 0 \Rightarrow \left[A \cancel{\sin(0)}^{0} + B \cancel{\cos(0)}^{1}\right] \cos(\omega t+\phi) = 0$

$$\Rightarrow B\cos(\omega t+\phi) = 0$$

$$\Rightarrow B = 0$$

Thus <u>imposing a boundary condition</u> <u>restricts the type of solution</u> to $y(x,t) = A\sin(kx)\cos(\omega t+\phi)$

Quiz 2

The other boundary conditions requires:

$$y(L,t) = 0 \Rightarrow A\sin(kL)\cos(\omega t+\phi) = 0$$

$$\Rightarrow \sin(kL) = 0$$

$$\Rightarrow kL = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\ldots, n\pi, \ldots$$

↳ any integer.

We see that

Imposing both boundary conditions restricts possible values of k. For a string with fixed ends the values of k can be indexed by an integer, $n = 0, 1, 2, \ldots$ and gives. $k_n = \frac{n\pi}{L}$

Quiz 3

Using $k = \omega/v \Rightarrow \omega = kv$ we see that there are only certain frequencies possible. Since k is indexed by n, so is ω. The possible frequencies are

$$\boxed{\omega_n = \frac{n\pi}{L} v \qquad n = 0, 1, 2, \ldots}$$

To summarize.

The string satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = v^{-2}\frac{\partial^2 y}{\partial x^2}$$

subject to boundary values $y(0,t)=0$ and $y(L,t)=0$

$\Downarrow$

Separation of variables gives a general form of solution

$$y(x,t) = \left[A\sin(kx) + B\cos(kx)\right]\cos(\omega t+\phi)$$

where $\omega = kv$

$\Downarrow$

Imposing boundary conditions gives the following standing wave solutions:

$$y_n(x,t) = A_n \sin(k_n x)\cos(\omega_n t + \phi_n)$$

where $n = 1,2,3,\ldots$ and

$$k_n = \frac{n\pi}{L} \qquad \omega_n = \frac{n\pi}{L}v$$

and A_n, ϕ_n are constants.

These solutions are called normal modes. Then:

1) normal modes are indexed by integers
2) each has a distinct frequency + wavenumber
3) the disturbance does not propagate left or right

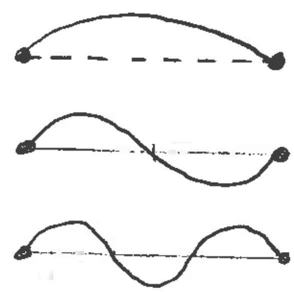
The wavelength of the n^{th} mode is given by:

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L} \Rightarrow \quad \lambda_n = \frac{2L}{n}$$

The frequency is $\quad 2\pi f_n = \frac{n\pi}{L} v \quad \Rightarrow \quad f_n = \frac{nv}{2L}$.

Snapshots at $t=0$ with $\phi_n = 0$ are illustrated

n	$y_n(x,0)$	λ_n	f_n
1	$A_1 \sin\left(\frac{\pi x}{L}\right)$	$2L$	$\frac{v}{2L}$
2	$A_2 \sin\left(\frac{2\pi x}{L}\right)$	L	$\frac{v}{2L}\,2$
3	$A_3 \sin\left(\frac{3\pi x}{L}\right)$	$2L/3$	$\frac{v}{2L}\,3$



Demo: Falstad animation - show normal modes
- loaded string
- damping $\rightarrow 0$
tension $\rightarrow$ lower.
set stopped $\rightarrow$ harmonics

Read 6.2